

**Figure 10.1** General features for linearized supersonic flow past a thin airfoil.

is isentropic. Thus, the isentropic relations developed in Chapter 8 can be used to describe the subsequent acceleration of the flow around the airfoil.

Experience has shown that the leading edge and the trailing edge of supersonic airfoils should be sharp (or only slightly rounded) and the section relatively thin. If the leading edge is not sharp (or only slightly rounded), the leading-edge shock wave will be detached and relatively strong, causing relatively large wave drag. We shall consider, therefore, profiles of the general cross section shown in Fig. 10.1. For these thin airfoils at relatively small angles of attack, we can apply the method of small perturbations to obtain theoretical approximations to the aerodynamic characteristics of the two-dimensional airfoils.

By “thin” airfoil, we mean that the thickness, camber, and angle of attack of the section are such that the local flow direction at the airfoil surface deviates only slightly from the free-stream direction. First we treat the Ackeret, or linearized, theory for thin airfoils and then higher-order theories. The coefficients calculated using the linearized and higher-order theories will be compared with the values calculated using the techniques of Chapter 8.

## 10.1 LINEAR THEORY

The basic assumption of linear theory is that pressure waves generated by thin sections are sufficiently weak that they can be treated as Mach waves. Under this assumption, the flow is isentropic everywhere. The pressure and the velocity changes for a small expansive change in flow direction (an acceleration) have already been derived in Chapter 8 [i.e., equation (8.55)]. Let us define the free-stream flow direction to be given by  $\theta_\infty = 0$ . For small changes in  $\theta$ , we can use equation (8.55) to calculate the change in pressure:

$$p - p_\infty = -\rho_\infty U_\infty (U - U_\infty) \quad (10.1a)$$

$$\frac{U_\infty - U}{U_\infty} = \frac{\theta}{\sqrt{M_\infty^2 - 1}} \quad (10.1b)$$

We will define the angle  $\theta$  so that we obtain the correct sign for the pressure coefficient both for left-running characteristics and for right-running characteristics. Combining these relations yields

$$C_p = + \frac{2\theta}{\sqrt{M_\infty^2 - 1}} \quad (10.1c)$$

which can be used to calculate the pressure on the airfoil surface, since  $\theta$  is known at every point on the airfoil surface.

A positive pressure coefficient is associated with a compressive change in flow direction relative to the free-stream flow. If the flow is turned toward the upstream Mach waves, the local pressure coefficient is positive and is greatest where the local inclination is greatest. Thus, for the double-convex-arc airfoil section shown in Fig. 10.1, the pressure is greatest at the leading edge, being greater on the lower surface when the airfoil is at a positive angle of attack. Flow accelerates continuously from the leading edge to the trailing edge for both the lower surface and the upper surface. The pressure coefficient is zero (i.e., the local static pressure is equal to the free-stream value) at those points where the local surface is parallel to the free stream. Downstream, the pressure coefficient is negative, which corresponds to an expansive change in flow direction.

The pressure coefficients calculated using the linearized approximation and Busemann's second-order approximation (to be discussed in the next section) are compared in Fig. 10.2 with the exact values of Prandtl-Meyer theory for expansive turns and of oblique shock-wave theory for compressive turns. For small deflections, linear theory provides suitable values for engineering calculations.

Since the slope of the surface of the airfoil section measured with respect to the free-stream direction is small, we can set it equal to its tangent. Referring to Fig. 10.3, we can write

$$\theta_u = \frac{dz_u}{dx} - \alpha \quad (10.2a)$$

$$\theta_l = -\frac{dz_l}{dx} + \alpha \quad (10.2b)$$

The lift, the drag, and the moment coefficients for the section can be determined using equations (10.1c) and (10.2).

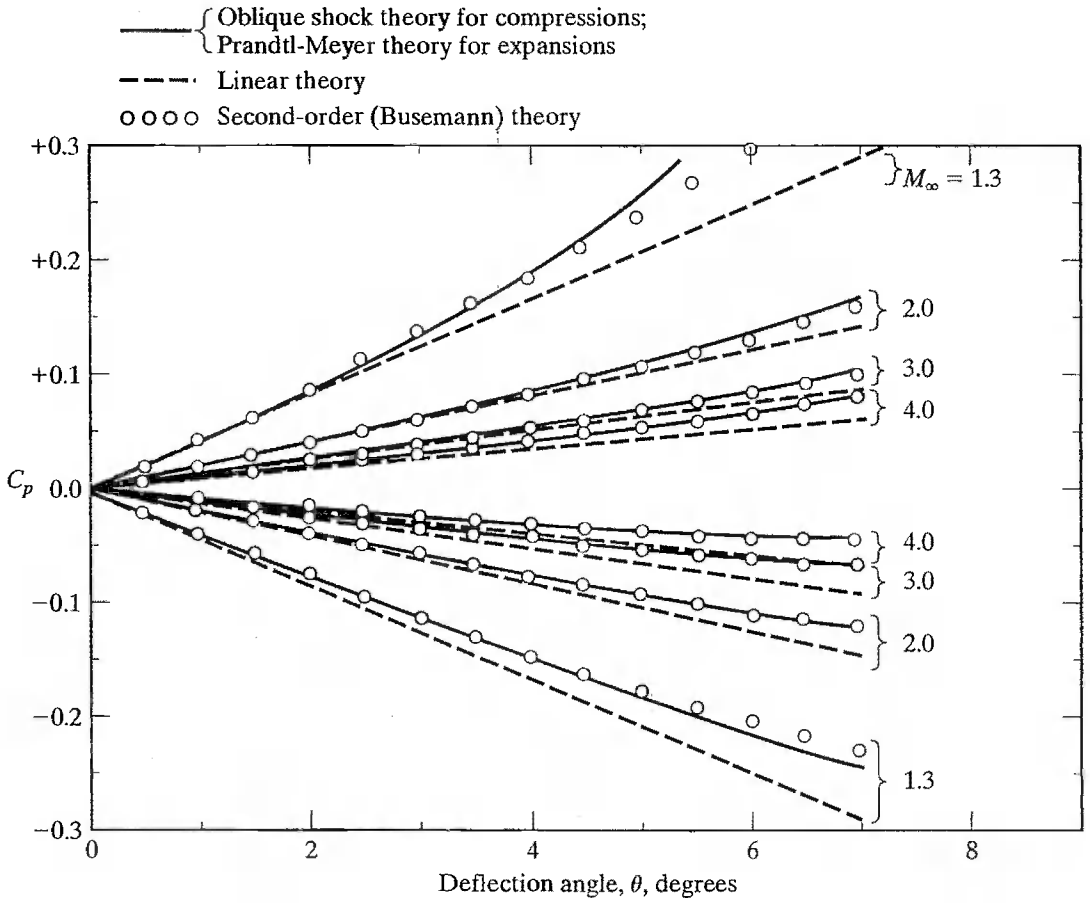
### 10.1.1 Lift

Referring to Fig. 10.4, we see that the incremental lift force (per unit span) acting on the chordwise segment  $ABCD$  of the airfoil section is

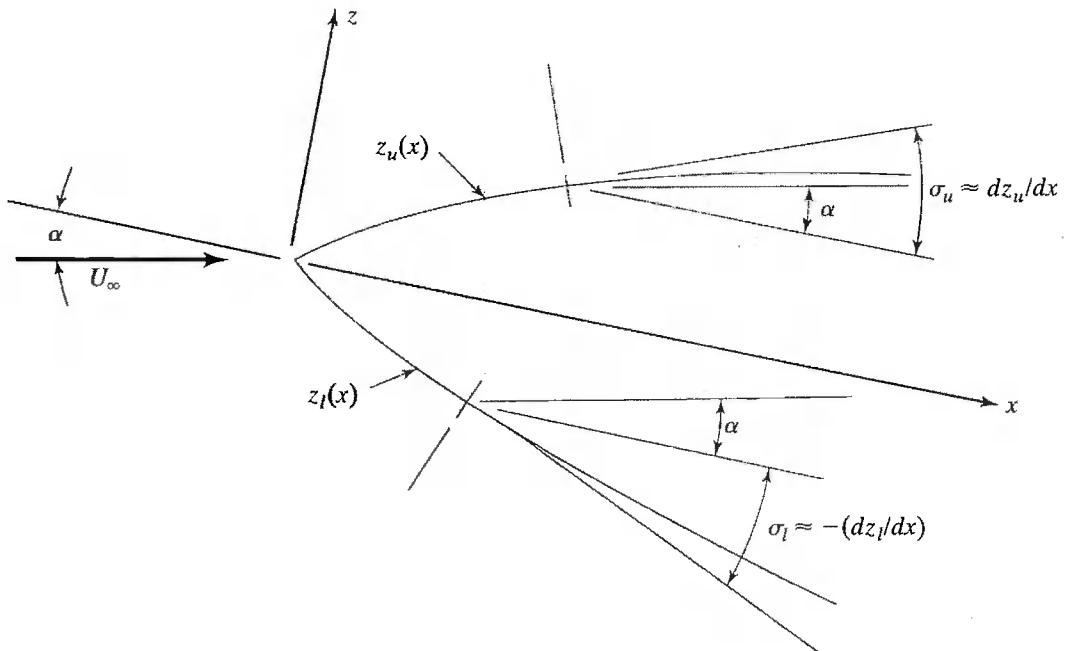
$$dl = p_l ds_l \cos \theta_l - p_u ds_u \cos \theta_u \quad (10.3)$$

Employing the usual thin-airfoil assumptions, equation (10.3) can be written as

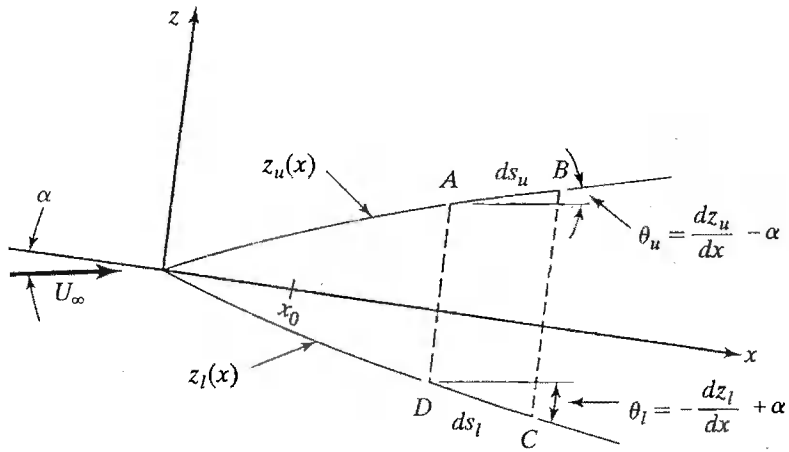
$$dl \approx (p_l - p_u) dx \quad (10.4)$$



**Figure 10.2** Theoretical pressure coefficients as a function of the deflection angle (relative to the stream) for various two-dimensional theories.



**Figure 10.3** Detailed sketch of an airfoil section.



**Figure 10.4** Thin-airfoil geometry for determining  $C_l$ ,  $C_d$ , and  $C_{m,x_0}$ .

In coefficient form, we have

$$dC_l \approx (C_{pl} - C_{pu})d\left(\frac{x}{c}\right) \tag{10.5}$$

Using equations (10.1c) and (10.2), equation (10.5) becomes

$$dC_l = \frac{2}{\sqrt{M_\infty^2 - 1}} \left( 2\alpha - \frac{dz_l}{dx} - \frac{dz_u}{dx} \right) d\left(\frac{x}{c}\right) \tag{10.6}$$

where, without loss of generality, we have assumed that positive values both for  $\theta_u$  and  $\theta_l$  represent compressive changes in the flow direction from the free-stream flow.

We can calculate the total lift of the section by integrating equation (10.6) from  $x/c = 0$  to  $x/c = 1$ . Note that since the  $z_u = z_l = 0$  at both the leading edge and the trailing edge,

$$\int_0^1 \frac{dz_l}{dx} d\left(\frac{x}{c}\right) = 0 \tag{10.7a}$$

and

$$\int_0^1 \frac{dz_u}{dx} d\left(\frac{x}{c}\right) = 0 \tag{10.7b}$$

Thus,

$$C_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \tag{10.8}$$

We see that, in the linear approximation for supersonic flow past a thin airfoil, the lift coefficient is independent of the camber and of the thickness distribution. Furthermore, the angle of attack for zero lift is zero. The lift-curve slope is seen to be only a function of the free-stream Mach number, since

$$\frac{dC_l}{d\alpha} = \frac{4}{\sqrt{M_\infty^2 - 1}} \tag{10.9}$$

Examining equation (10.9), we see that for  $M_\infty \gtrsim 1.185$ , the lift-curve slope is less than the theoretical value for incompressible flow past a thin airfoil, which is  $2\pi$  per radian.

### 10.1.2 Drag

The incremental drag force due to the inviscid flow acting on the arbitrary chordwise element  $ABCD$  of Fig. 10.4 is

$$dd = p_l ds_l \sin \theta_l + p_u ds_u \sin \theta_u \quad (10.10)$$

Again, using the assumptions common to small deflection angles, equation (10.10) becomes

$$dd = p_l \theta_l dx + p_u \theta_u dx \quad (10.11)$$

In coefficient form, we have

$$dC_d = (C_{pl}\theta_l + C_{pu}\theta_u)d\left(\frac{x}{c}\right) + \frac{2}{\gamma M_\infty^2}(\theta_l + \theta_u)d\left(\frac{x}{c}\right) \quad (10.12)$$

Using equation (10.1c) for compressive turns and equation (10.2) to approximate the angles, equation (10.12) yields

$$\begin{aligned} dC_d = & \frac{2}{\sqrt{M_\infty^2 - 1}} \left[ 2\alpha^2 + \left(\frac{dz_u}{dx}\right)^2 + \left(\frac{dz_l}{dx}\right)^2 \right] d\left(\frac{x}{c}\right) \\ & + \left[ \frac{-4\alpha}{\sqrt{M_\infty^2 - 1}} \left(\frac{dz_l}{dx} + \frac{dz_u}{dx}\right) + \frac{2}{\gamma M_\infty^2} \left(\frac{dz_u}{dx} - \frac{dz_l}{dx}\right) \right] d\left(\frac{x}{c}\right) \end{aligned} \quad (10.13)$$

Using equation (10.7), we find that the integration of equation (10.13) yields

$$C_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} + \frac{2}{\sqrt{M_\infty^2 - 1}} \int_0^1 \left[ \left(\frac{dz_u}{dx}\right)^2 + \left(\frac{dz_l}{dx}\right)^2 \right] d\left(\frac{x}{c}\right) \quad (10.14)$$

Note that for the small-angle assumptions commonly used in analyzing flow past a thin airfoil,

$$\frac{dz_u}{dx} = \tan \sigma_u \approx \sigma_u$$

Thus,

$$\frac{1}{c} \int_0^c \left(\frac{dz_u}{dx}\right)^2 dx = \overline{\sigma_u^2} \quad (10.15a)$$

Similarly, we can write

$$\frac{1}{c} \int_0^c \left(\frac{dz_l}{dx}\right)^2 dx = \overline{\sigma_l^2} \quad (10.15b)$$

We can use these relations to replace the integrals of equation (10.14) by the average values that they represent. Thus, the section-drag coefficient for this frictionless flow model is

$$C_d = \frac{d}{q_\infty c} = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} + \frac{2}{\sqrt{M_\infty^2 - 1}}(\overline{\sigma_u^2} + \overline{\sigma_l^2}) \quad (10.16)$$

Note that the drag is not zero even though the airfoil is of infinite span and the viscous forces have been neglected. This drag component, which is not present in subsonic flows, is known as *wave drag*. Note also that, as this small perturbation solution shows, it is not necessary that shock waves be present for wave drag to exist. Such was also the case in Example 8.3, which examined the shock-free flow past an infinitesimally thin, parabolic arc airfoil.

Let us examine the character of the terms in equation (10.16). Since the lift is directly proportional to the angle of attack and is independent of the section thickness, the first term is called the *wave drag due to lift* or the *induced wave drag* and is independent of the shape of the airfoil section. The second term is often referred to as the *wave drag due to thickness* and depends only on the shape of the section. Equation (10.16) also indicates that, for a given configuration, the wave-drag coefficient decreases with increasing Mach number. If we were to account for the effects of viscosity, we could write

$$C_d = C_{d, \text{ due to lift}} + C_{d, \text{ thickness}} + C_{d, \text{ friction}} \quad (10.17a)$$

where

$$C_{d, \text{ due to lift}} = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} = \alpha C_l \quad (10.17b)$$

and

$$C_{d, \text{ thickness}} = \frac{2}{\sqrt{M_\infty^2 - 1}}(\overline{\sigma_u^2} + \overline{\sigma_l^2}) \quad (10.17c)$$

Note also that  $C_{d, \text{ thickness}}$  is the  $C_{d0}$  of previous chapters.

### 10.1.3 Pitching Moment

Let us now use linear theory to obtain an expression for the pitching moment coefficient. Referring to Fig. 10.4, the incremental moment (taken as positive for nose up relative to the free stream) about the arbitrary point  $x_0$  on the chord is

$$dm_{x_0} = (p_u - p_l)(x - x_0)dx \quad (10.18)$$

where we have incorporated the usual small-angle assumptions and have neglected the contributions of the chordwise components of  $p_u$  and  $p_l$  to the pitching moment.

In coefficient form, we have

$$dC_{m_{x_0}} = (C_{p_u} - C_{p_l}) \frac{x - x_0}{c} d\left(\frac{x}{c}\right) \quad (10.19)$$

Substituting equation (10.1c) into equation (10.19) yields

$$dC_{m_{x_0}} = \frac{2}{\sqrt{M_\infty^2 - 1}}(\theta_u - \theta_l) \frac{x - x_0}{c} d\left(\frac{x}{c}\right) \quad (10.20)$$

Substituting equation (10.2) into equation (10.20) and integrating along the chord gives

$$C_{m_{x_0}} = \frac{-4\alpha}{\sqrt{M_\infty^2 - 1}} \left( \frac{1}{2} - \frac{x_0}{c} \right) + \frac{2}{\sqrt{M_\infty^2 - 1}} \int_0^1 \left( \frac{dz_u}{dx} + \frac{dz_l}{dx} \right) \frac{x - x_0}{c} d\left(\frac{x}{c}\right) \quad (10.21)$$

Note that the average of the upper surface coordinate  $z_u$  and the lower surface coordinate  $z_l$  defines the mean camber coordinate  $z_c$ ,

$$\frac{1}{2}(z_u + z_l) = z_c.$$

We can then write equation (10.21) as

$$C_{m_{x_0}} = \frac{-4\alpha}{\sqrt{M_\infty^2 - 1}} \left( \frac{1}{2} - \frac{x_0}{c} \right) + \frac{4}{\sqrt{M_\infty^2 - 1}} \int_0^1 \frac{dz_c}{dx} \frac{x - x_0}{c} d\left(\frac{x}{c}\right) \quad (10.22)$$

where we have assumed that  $z_u = z_l$  both at the leading edge and at the trailing edge.

As discussed in Section 5.4.2, the aerodynamic center is that point about which the pitching moment coefficient is independent of the angle of attack. It may also be considered to be that point along the chord at which all changes in lift effectively take place. Thus, equation (10.22) shows that the aerodynamic center is at midchord for a thin airfoil in a supersonic flow. This is in contrast to the thin airfoil in an incompressible flow where the aerodynamic center is at the quarter chord.

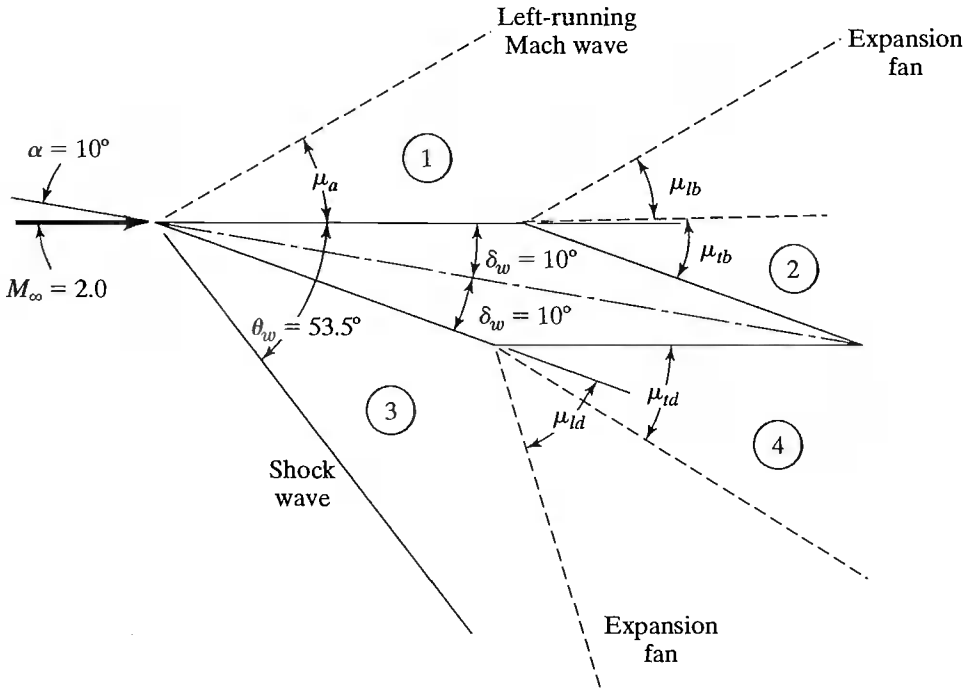
**EXAMPLE 10.1:** Use linear theory to calculate the lift coefficient, the wave-drag coefficient, and the pitching-moment coefficient

Let us use the linear theory to calculate the lift coefficient, the wave-drag coefficient, and the pitching-moment coefficient for the airfoil section whose geometry is illustrated in Fig 10.5. For purposes of discussion, the flow field has been divided into numbered regions, which correspond to each of the facets of the double-wedge airfoil, as shown. In each region the flow properties are such that the static pressure and the Mach number are constant, although they differ from region to region. We seek the lift coefficient, the drag coefficient, and the pitching-moment coefficient per unit span of the airfoil given the free-stream flow conditions, the angle of attack, and the geometry of the airfoil neglecting the effect of the viscous boundary layer. The only forces acting on the airfoil are the pressure forces. Therefore, once we have determined the static pressure in each region, we can then integrate to find the resultant forces and moments.

**Solution:** Let us now evaluate the various geometric parameters required for the linearized theory:

$$z_u(x) = \begin{cases} x \tan 10^\circ & \text{for } 0 \leq x \leq \frac{c}{2} \\ (c - x) \tan 10^\circ & \text{for } \frac{c}{2} \leq x \leq c \end{cases}$$

$$z_l(x) = \begin{cases} -x \tan 10^\circ & \text{for } 0 \leq x \leq \frac{c}{2} \\ -(c - x) \tan 10^\circ & \text{for } \frac{c}{2} \leq x \leq c \end{cases}$$



**Figure 10.5** Wave pattern for a double-wedge airfoil in a Mach 2 stream.

Furthermore,

$$\overline{\sigma_l^2} = \int_0^1 \sigma_l^2 d\left(\frac{x}{c}\right) = \delta_w^2$$

and

$$\overline{\sigma_u^2} = \int_0^1 \sigma_u^2 d\left(\frac{x}{c}\right) = \delta_w^2$$

We can use equation (10.8) to calculate the section lift coefficient for a 10° angle of attack at  $M_\infty = 2.0$ :

$$C_l = \frac{4(10\pi/180)}{\sqrt{2^2 - 1}} = 0.4031$$

Similarly, the drag coefficient can be calculated using equation (10.16):

$$C_d = \frac{4(10\pi/180)^2}{\sqrt{2^2 - 1}} + \frac{2}{\sqrt{2^2 - 1}} \left[ \left( \frac{10}{57.296} \right)^2 + \left( \frac{10}{57.296} \right)^2 \right]$$

Therefore,

$$C_d = 0.1407$$

The lift/drag ratio is

$$\frac{l}{d} = \frac{C_l}{C_d} = \frac{0.4031}{0.1407} = 2.865$$



Note that in order to clearly illustrate the calculation procedures, this airfoil section is much thicker (i.e.,  $t = 0.176c$ ) than typical supersonic airfoil sections for which  $t \approx 0.05c$  (see Table 5.1). The result is a relatively low lift/drag ratio.

Similarly, equation (10.22) for the pitching moment coefficient gives

$$C_{m_{x_0}} = \frac{-4\alpha}{\sqrt{M_\infty^2 - 1}} \left( \frac{1}{2} - \frac{x_0}{c} \right)$$

since the mean-camber coordinate  $x_c$  is everywhere zero. At midchord, we have

$$C_{m_{0.5c}} = 0$$

This is not a surprising result, since equation (10.22) indicates that the moment about the aerodynamic center of a symmetric (zero camber) thin airfoil in supersonic flow vanishes.

## 10.2 SECOND-ORDER THEORY (BUSEMANN'S THEORY)

Equation (10.1c) is actually the first term in a Taylor series expansion of  $\Delta p$  in powers of  $\theta$ . Busemann showed that a more accurate expression for the pressure change resulted if the  $\theta^2$  term were retained in the expansion. His result [given in Edmondson, et al. (1945)] in terms of the pressure coefficient is

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} + \left[ \frac{(\gamma + 1)M_\infty^4 - 4M_\infty^2 + 4}{2(M_\infty^2 - 1)^2} \right] \theta^2 \quad (10.23a)$$

or

$$C_p = C_1\theta + C_2\theta^2 \quad (10.23b)$$

Again,  $\theta$  is positive for a compression turn and negative for an expansion turn. We note that the  $\theta^2$  term in equation (10.23) is always a positive contribution. Table 10.1 gives  $C_1$  and  $C_2$  for various Mach numbers in air.

It is important to note that since the pressure waves are treated as Mach waves, the turning angles must be small. These assumptions imply that the flow is isentropic everywhere. Thus, equations (10.5), (10.12), and (10.19), along with equation (10.23), can still be used to find  $C_l$ ,  $C_d$ , and  $C_{m_{x_0}}$ . Note that Fig. 10.2 shows that Busemann's theory agrees even more closely with the results obtained from oblique shock and Prandtl-Meyer expansion theory than does linear theory.

### EXAMPLE 10.2: Use Busemann's theory to calculate the lift coefficient, the wave-drag coefficient, and the pitching-moment coefficient

Let us calculate the pressure coefficient on each panel of the airfoil in Example 10.1 using Busemann's theory. We will use equation (10.23).

**TABLE 10.1** Coefficients  $C_1$  and  $C_2$  for the Busemann Theory for Perfect Air,  $\gamma = 1.4$

$M_\infty$	$C_1$	$C_2$
1.10	4.364	30.316
1.12	3.965	21.313
1.14	3.654	15.904
1.16	3.402	12.404
1.18	3.193	10.013
1.20	3.015	8.307
1.22	2.862	7.050
1.24	2.728	6.096
1.26	2.609	5.356
1.28	2.503	4.771
1.30	2.408	4.300
1.32	2.321	3.916
1.34	2.242	3.599
1.36	2.170	3.333
1.38	2.103	3.109
1.40	2.041	2.919
1.42	1.984	2.755
1.44	1.930	2.614
1.46	1.880	2.491
1.48	1.833	2.383
1.50	1.789	2.288
1.52	1.747	2.204
1.54	1.708	2.129
1.56	1.670	2.063
1.58	1.635	2.003
1.60	1.601	1.949
1.70	1.455	1.748
1.80	1.336	1.618
1.90	1.238	1.529
2.00	1.155	1.467
2.50	0.873	1.320
3.00	0.707	1.269
3.50	0.596	1.248
4.00	0.516	1.232
5.00	0.408	1.219
10.0	0.201	1.204
$\infty$	0	1.200

**Solution:** Since panel 1 is parallel to the free stream,  $C_{p1} = 0$  as before. For panel 2,

$$\begin{aligned}
 C_{p2} &= \frac{2(-20\pi/180)}{\sqrt{2^2 - 1}} + \frac{(1.4 + 1)(2)^4 - 4(2)^2 + 4\left(\frac{20\pi}{180}\right)^2}{2(2^2 - 1)^2} \\
 &= -0.4031 + 0.1787 \\
 &= -0.2244
 \end{aligned}$$

As noted, the airfoil in this sample problem is relatively thick, and therefore the turning angles are quite large. As a result, the differences between linear theory and higher-order approximations are significant but not unexpected. For panel 3,

$$\begin{aligned} C_{p3} &= 0.4031 + 0.1787 \\ &= 0.5818 \end{aligned}$$

For panel 4,

$$C_{p4} = 0$$

since the flow along surface 4 is parallel to the free stream.

Having determined the pressures acting on the individual facets of the double-wedge airfoil, let us now determine the section lift coefficient:

$$C_l = \frac{\sum p \cos \theta (0.5c / \cos \delta_w)}{(\gamma/2) p_\infty M_\infty^2 c} \quad (10.24a)$$

where  $\delta_w$  is the half-angle of the double-wedge configuration. We can use the fact that the net force in any direction due to a constant pressure acting on a closed surface is zero to get

$$C_l = \frac{1}{2 \cos \delta_w} \sum C_p \cos \theta \quad (10.24b)$$

where the signs assigned to the  $C_p$  terms account for the direction of the force. Thus,

$$\begin{aligned} C_l &= \frac{1}{2 \cos 10^\circ} (-C_{p2} \cos 20^\circ + C_{p3} \cos 20^\circ) \\ &= 0.3846 \end{aligned}$$

Similarly, we can calculate the section wave-drag coefficient:

$$C_d = \frac{\sum p \sin \theta (0.5c / \cos \delta_w)}{(\gamma/2) p_\infty M_\infty^2 c} \quad (10.25a)$$

or

$$C_d = \frac{1}{2 \cos \delta_w} \sum C_p \sin \theta \quad (10.25b)$$

Applying this relation to the airfoil section of Fig 10.5, the section wave-drag coefficient for  $\alpha = 10^\circ$  is

$$\begin{aligned} C_d &= \frac{1}{2 \cos 10^\circ} (C_{p3} \sin 20^\circ - C_{p2} \sin 20^\circ) \\ &= 0.1400 \end{aligned}$$

Let us now calculate the moment coefficient with respect to the midchord of the airfoil section (i.e., relative to  $x = 0.5c$ ). As we have seen, the theoretical solutions for linearized flow show that the midchord point is the aerodynamic center for a thin airfoil in a supersonic flow. Since the pressure is constant on each of the facets of the double-wedge airfoil of Fig. 10.5 (i.e., in each numbered region), the force acting on a given facet will be normal to the surface and will act at the midpoint of the panel. Thus,

$$\begin{aligned}
 C_{m_{0.5c}} = \frac{m_{0.5c}}{\frac{1}{2}\rho_{\infty}U_{\infty}^2c^2} &= (-p_1 + p_2 + p_3 - p_4)\frac{c^2/8}{\frac{1}{2}\rho_{\infty}U_{\infty}^2c^2} \\
 &+ (p_1 - p_2 - p_3 + p_4)\frac{(c^2/8)\tan^2\delta_w}{\frac{1}{2}\rho_{\infty}U_{\infty}^2c^2} \quad (10.26)
 \end{aligned}$$

Note that, as usual, a nose-up pitching moment is considered positive. Also note that we have accounted for terms proportional to  $\tan^2\delta_w$ . Since the pitching moment due to a uniform pressure acting on any closed surface is zero, equation (10.26) can be written as

$$\begin{aligned}
 C_{m_{0.5c}} &= (-C_{p1} + C_{p2} + C_{p3} - C_{p4})\frac{1}{8} \\
 &+ (C_{p1} - C_{p2} - C_{p3} + C_{p4})\frac{\tan^2\delta_w}{8} \quad (10.27)
 \end{aligned}$$

The reader is referred to equations (5.11) through (5.15) for a review of the technique. Thus,

$$C_{m_{0.5c}} = 0.04329$$


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### 10.3 SHOCK-EXPANSION TECHNIQUE

The techniques discussed thus far assume that compressive changes in flow direction are sufficiently small that the inviscid flow is everywhere isentropic. In reality, a shock wave is formed as the supersonic flow encounters the two-dimensional double-wedge airfoil of the previous example problems: Since the shock wave is attached to the leading edge and is planar, the downstream flow is isentropic. Thus, the isentropic Prandtl-Meyer relations developed in Chapter 8 can be used to describe the acceleration of the flow around the airfoil. Let us use this shock-expansion technique to calculate the flow field around the airfoil shown in Fig. 10.5.

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**EXAMPLE 10.3:** Use the shock-expansion theory to calculate the lift coefficient, the wave-drag coefficient, and the pitching-moment coefficient.

For purposes of discussion, the flow field has been divided into numbered regions that correspond to each of the facets of the double-wedge airfoil, as shown in Fig. 10.5. As was true for the approximate theories, the flow properties in each region, such as the static pressure and the Mach number, are constant, although they differ from region to region. We will calculate the section lift coefficient, the section drag coefficient, and the section pitching moment coefficient for the inviscid flow.

**Solution:** Since the surface of region 1 is parallel to the free stream, the flow does not turn in going from the free-stream conditions ( $\infty$ ) to region 1. Thus, the

properties in region 1 are the same as in the free stream. The pressure coefficient on the airfoil surface bounding region 1 is zero. Thus,

$$M_1 = 2.0 \quad \nu_1 = 26.380^\circ \quad \theta_1 = 0^\circ \quad C_{p1} = 0.0$$

Furthermore, since the flow is not decelerated in going to region 1, a Mach wave (and not a shock wave) is shown as generated at the leading edge of the upper surface. Since the Mach wave is of infinitesimal strength, it has no effect on the flow. However, for completeness, let us calculate the angle between the Mach wave and the free-stream direction. The Mach angle is

$$\mu_a = \sin^{-1} \frac{1}{M_\infty} = 30^\circ$$

Since the surface of the airfoil in region 2 “turns away” from the flow in region 1, the flow accelerates isentropically in going from region 1 to region 2. To cross the left-running Mach waves dividing region 1 from region 2, we move along right-running characteristics. Therefore,

$$d\nu = -d\theta$$

Since the flow direction in region 2 is

$$\theta_2 = -20^\circ$$

$\nu_2$  is

$$\nu_2 = \nu_1 - \Delta\theta = 46.380^\circ$$

Therefore,

$$M_2 = 2.83$$

To calculate the pressure coefficient for region 2,

$$C_{p2} = \frac{p_2 - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{p_2 - p_\infty}{(\gamma/2)p_\infty M_\infty^2}$$

Thus,

$$C_{p2} = \left( \frac{p_2}{p_\infty} - 1 \right) \frac{2}{\gamma M_\infty^2}$$

Since the flow over the upper surface of the airfoil is isentropic,

$$p_{t\infty} = p_{t1} = p_{t2}$$

Therefore,

$$C_{p2} = \left( \frac{p_2}{p_{t2}} \frac{p_{t\infty}}{p_\infty} - 1 \right) \frac{2}{\gamma M_\infty^2}$$

Using Table 8.1. or equation (8.36), to calculate the pressure ratios given the values for  $M_\infty$  and for  $M_2$ ,

$$C_{p2} = \left( \frac{0.0352}{0.1278} - 1 \right) \frac{2}{1.4(4)} = -0.2588$$

The fluid particles passing from the free stream to region 3 are turned by the shock wave through an angle of  $20^\circ$ . The shock wave decelerates the flow and the pressure in region 3 is relatively high. To calculate the pressure coefficient for region 3, we must determine the pressure increase across the shock wave. Since we know that  $M_\infty = 2.0$  and  $\theta = 20^\circ$ , we can use Fig. 8.12b to find the value of  $C_{p3}$  directly; it is 0.66. As an alternative procedure, we can use Fig. 8.12 to find the shock-wave angle,  $\theta_w$ , which is equal to  $\theta_w = 53.5^\circ$ . Therefore, as discussed in Section 8.5, we can use  $M_\infty \sin \theta$  (instead of  $M_\infty$ ), which is 1.608, and the correlations of Table 8.3 to calculate the pressure increase across the oblique shock wave:

$$\frac{p_3}{p_\infty} = 2.848$$

Thus,

$$C_{p3} = \left( \frac{p_3}{p_\infty} - 1 \right) \frac{2}{\gamma M_\infty^2} = 0.66$$

Using Fig. 8.12c, we find that  $M_3 = 1.20$ .

Having determined the flow in region 3,

$$M_3 = 1.20 \quad \nu_3 = 3.558^\circ \quad \theta_3 = -20^\circ$$

one can determine the flow in region 4 using the Prandtl-Meyer relations. One crosses the right-running Mach waves dividing regions 3 and 4 on left-running characteristics. Thus,

$$d\nu = d\theta$$

Since  $\theta_4 = 0^\circ$ ,  $d\theta = +20^\circ$  and

$$\nu_4 = 23.558^\circ$$

so

$$M_4 = 1.90$$

Note that because of the dissipative effect of the shock wave, the Mach number in region 4 (whose surface is parallel to the free stream) is less than the free-stream Mach number.

Whereas the flow from region 3 to region 4 is isentropic, and

$$p_{t3} = p_{t4}$$

the presence of the shock wave causes

$$p_{t3} < p_{t\infty}$$

To calculate

$$\begin{aligned} C_{p4} &= \left( \frac{p_4}{p_\infty} - 1 \right) \frac{2}{\gamma M_\infty^2} \\ &= \left( \frac{p_4}{p_3} \frac{p_3}{p_\infty} - 1 \right) \frac{2}{\gamma M_\infty^2} \end{aligned}$$

the ratio

$$\frac{p_4}{p_3} = \frac{p_4}{p_{t4}} \frac{p_{t3}}{p_3}$$

can be determined since both  $M_3$  and  $M_4$  are known. The ratio  $p_3/p_\infty$  has already been found to be 2.848. Thus,

$$C_{p4} = \left[ \frac{0.1492}{0.4124} (2.848) - 1.0 \right] \frac{2}{1.4(4.0)} = 0.0108$$

We can calculate the section lift coefficient using equation (10.24).

$$\begin{aligned} C_l &= \frac{1}{2 \cos 10^\circ} (-C_{p1} - C_{p2} \cos 20^\circ + C_{p3} \cos 20^\circ + C_{p4}) \\ &= 0.4438 \end{aligned}$$

Similarly, using equation (10.25) to calculate the section wave-drag coefficient,

$$C_d = \frac{1}{2 \cos 10^\circ} (C_{p3} \sin 20^\circ - C_{p2} \sin 20^\circ)$$

Thus,

$$C_d = 0.1595$$

The lift/drag ratio in our example is

$$\frac{l}{d} = 2.782$$

for this airfoil section.

In some cases, it is of interest to locate the leading and trailing Mach waves of the Prandtl-Meyer expansion fans at  $b$  and  $d$ . Thus, using the subscripts  $l$  and  $t$  to indicate leading and trailing Mach waves, respectively, we have

$$\mu_{lb} = \sin^{-1} \frac{1}{M_1} = 30^\circ$$

$$\mu_{tb} = \sin^{-1} \frac{1}{M_2} = 20.7^\circ$$

$$\mu_{ld} = \sin^{-1} \frac{1}{M_3} = 56.4^\circ$$

$$\mu_{td} = \sin^{-1} \frac{1}{M_4} = 31.8^\circ$$

Each Mach angle is shown in Fig. 10.5.

To calculate the pitching moment about the midchord point, we substitute the values we have found for the pressure coefficients into equation (10.27) and get

$$C_{m_{0.5c}} = 0.04728$$

Example 10.3 illustrates how to calculate the aerodynamic coefficients using the *shock-expansion technique*. This approach is exact provided the relevant assumptions are satisfied. A disadvantage of the technique is that it is essentially a numerical method which does not give a closed-form solution for evaluating airfoil performance parameters, such as the section lift and drag coefficients. However, if the results obtained by the method applied to a variety of airfoils are studied, one observes that the most efficient airfoil sections for supersonic flow are thin with little camber and have sharp leading edges. (The reader is referred to Table 5.1 to see that these features are used on the high-speed aircraft.) Otherwise, wave drag becomes prohibitive.

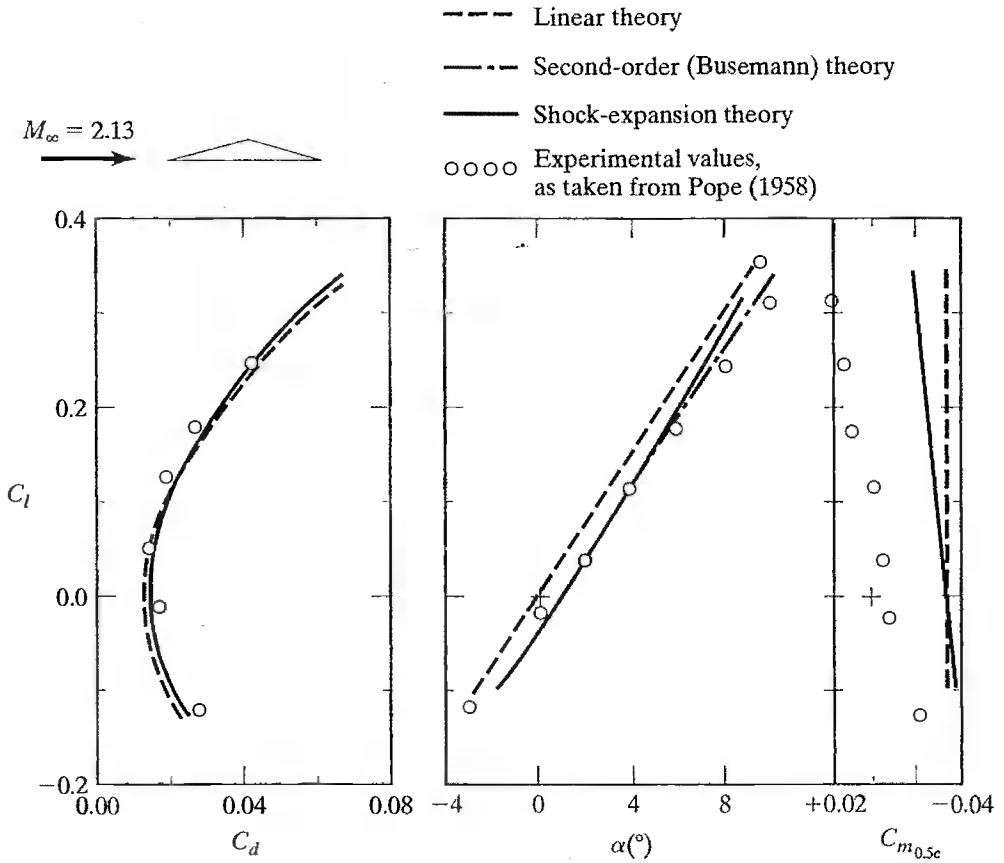
Thus, we have used a variety of techniques to calculate the inviscid flow field and the section aerodynamic coefficients for the double-wedge airfoil at an angle of attack of  $10^\circ$  in an airstream where  $M_\infty = 2.0$ . The theoretical values are compared in Table 10.2. Although the airfoil section considered in these sample problems is much thicker than those actually used on supersonic airplanes, there is reasonable agreement between the aerodynamic coefficients calculated using the various techniques. Thus, the errors in the local pressure coefficients tend to compensate for each other when the aerodynamic coefficients are calculated.

The theoretical values of the section aerodynamic coefficients as calculated using these three techniques are compared in Fig. 10.6, with experimental values taken from Pope (1958). The airfoil is reasonably thin and the theoretical values for the section lift coefficient and for the section wave-drag coefficient are in reasonable agreement with the data. The experimental values of the section moment coefficient exhibit the angle-of-attack dependence of the shock-expansion theory, but they differ in magnitude. Note that, for the airfoil shown in Fig. 10.6,  $C_l$  is negative at zero angle of attack. This is markedly different from the subsonic result, where the section lift coefficient is positive for a cambered airfoil at zero angle of attack. This is another example illustrating that the student should not apply intuitive ideas for subsonic flow to supersonic flows.

**TABLE 10.2** Comparison of the Aerodynamic Parameters for the Two-Dimensional Airfoil Section of Fig. 10.5,  $M_\infty = 2.0$ ,  $\alpha = 10^\circ$

	<i>Linearized (Ackeret) theory</i>	<i>Second-order (Busemann) theory</i>	<i>Shock- expansion technique</i>
$C_{p1}$	0.0000	0.0000	0.0000
$C_{p2}$	-0.4031	-0.2244	-0.2588
$C_{p3}$	+0.4031	+0.5818	+0.660
$C_{p4}$	0.0000	0.0000	+0.0108
$C_l$	0.4031	0.3846	0.4438
$C_d$	0.1407	0.1400	0.1595
$C_{m_{0.5c}}$	0.0000	0.04329	0.04728

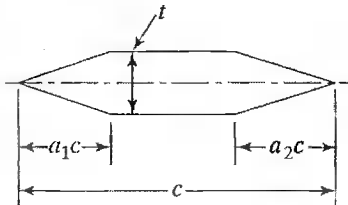




**Figure 10.6** Comparison of the experimental and the theoretical values of  $C_l$ ,  $C_d$ , and  $C_{m_{0.5c}}$ , for supersonic flow past an airfoil.

**PROBLEMS**

**10.1.** Consider supersonic flow past the thin airfoil shown in Fig. P10.1. The airfoil is symmetric about the chord line. Use linearized theory to develop expressions for the lift coefficient, the drag coefficient, and the pitching-moment coefficient about the midchord. The resultant expressions should include the free-stream Mach number, the constants  $a_1$  and  $a_2$ , and the thickness ratio  $t/c$  ( $\equiv \tau$ ). Show that, for a fixed thickness ratio, the wave drag due to thickness is a minimum when  $a_1 = a_2 = 0.5$ .



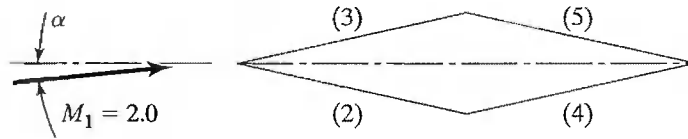
**Figure P10.1**

**10.2.** Consider the infinitesimally thin airfoil which has the shape of a parabola:

$$x^2 = -\frac{c^2}{z_{\max}}(z - z_{\max})$$

The leading edge of the profile is tangent to the direction of the oncoming airstream. This is the airfoil of Example 8.3. Use linearized theory for the following.

- (a) Find the expressions for the lift coefficient, the drag coefficient, the lift/drag ratio, and the pitching-moment coefficient about the leading edge. The resultant expressions should include the free-stream Mach number and the parameter,  $z_{\max}/c$ .
  - (b) Graph the pressure coefficient distribution as a function of  $x/c$  for the upper surface and for the lower surface. The calculations are for  $M_\infty = 2.059$  and  $z_{\max} = 0.1c$ . Compare the pressure distributions for linearized theory with those of Example 8.3.
  - (c) Compare the lift coefficient and the drag coefficient calculated using linearized theory for  $M_\infty = 2.059$  and for  $z_{\max} = 0.1c$  with those calculated in Example 8.3.
- 10.3.** Consider the double-wedge profile airfoil shown in Fig. P10.3. If the thickness ratio is 0.04 and the free-stream Mach number is 2.0 at an altitude of 12 km, use linearized theory to compute the lift coefficient, the drag coefficient, the lift/drag ratio, and the moment coefficient about the leading edge. Also, compute the static pressure and the Mach number in each region of the sketch. Make these calculations for angles of attack of  $2.29^\circ$  and  $5.0^\circ$ .



**Figure P10.3**

- 10.4.** Repeat Problem 10.3 using second-order (Busemann) theory.
- 10.5.** Repeat Problem 10.3 using the shock-expansion technique. What is the maximum angle of attack at which this airfoil can be placed and still generate a weak shock wave?
- 10.6.** Calculate the lift coefficient, the drag coefficient, and the coefficient for the moment about the leading edge for the airfoil of Problem 10.3 and for the same angles of attack, if the flow were incompressible subsonic.
- 10.7.** Verify the theoretical correlations presented in Fig. 10.6. Note that for this airfoil section,

$$\tau = \frac{t}{c} = 0.063$$

Furthermore, the free-stream Mach number is 2.13.

- 10.8.** For linearized theory, it was shown [i.e., equation (10.17c)] that the section drag coefficient due to thickness is

$$C_{d, \text{thickness}} = \frac{2}{\sqrt{M_1^2 - 1}} (\overline{\sigma_u^2} + \overline{\sigma_l^2})$$

If  $\tau$  is the thickness ratio, show that

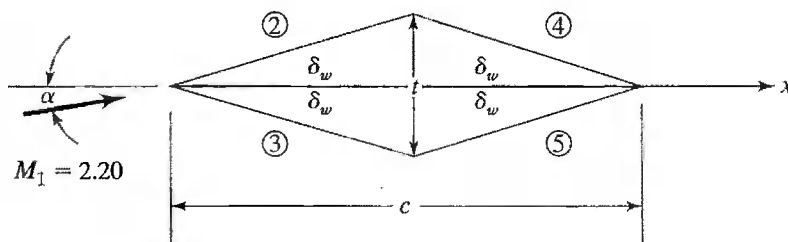
$$C_{d, \text{thickness}} = \frac{4\tau^2}{\sqrt{M_1^2 - 1}}$$

for a symmetric, double-wedge airfoil section and that

$$C_{d, \text{thickness}} = \frac{5.33\tau^2}{\sqrt{M_1^2 - 1}}$$

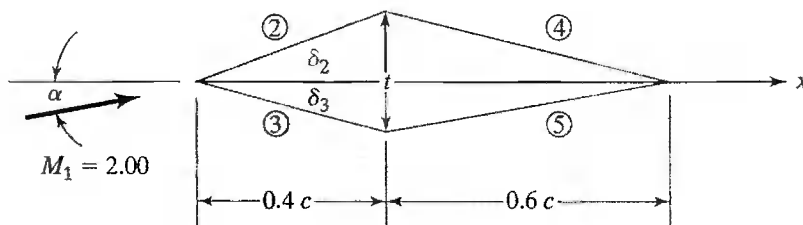
for a biconvex airfoil section. In doing this problem, we are verifying the values for  $K_1$  given in Table 11.1.

- 10.9.** Consider the symmetric, diamond-shaped airfoil section (Fig. P10.9; as shown in the sketch, all four facets have the same value of  $\delta_w$ ) exposed to a Mach 2.20 stream in a wind tunnel. For the wind tunnel,  $p_{t1} = 125$  psia  $T_t = 600^\circ\text{R}$ . The airfoil is such that the maximum thickness  $t$  equals to  $0.07c$ . The angle of attack is  $6^\circ$ .
- Using the shock-wave relations where applicable and the isentropic expansion relations (Prandtl-Meyer) where applicable, calculate the pressures in regions 2 through 5.
  - Using the linear-theory relations, calculate the pressures in regions 2 through 5.
  - Calculate  $C_A$ ,  $C_N$ ,  $C_d$ , and  $C_{m_{0.5c}}$  for this configuration.  $C_A$  is the axial force coefficient for the force along the axis (i.e., parallel to the chordline of the airfoil) and  $C_N$  is normal force coefficient (i.e., normal to the chordline of the airfoil).



**Figure P10.9**

- 10.10.** Consider the “cambered,” diamond-shaped airfoil exposed to a Mach 2.00 stream in a wind tunnel (Fig. P10.10). For the wind tunnel,  $p_{t1} = 125$  psia  $T_t = 650^\circ\text{R}$ . The airfoil is such that  $\delta_2 = 8^\circ$ ,  $\delta_3 = 2^\circ$ ; the maximum thickness  $t$  equals to  $0.07c$  and is located at  $x = 0.40c$ . The angle of attack is  $10^\circ$ .
- Using the shock-wave relations where applicable and the isentropic expansion relations (Prandtl-Meyer) where applicable, calculate the pressures in regions 2 through 5.
  - Using the linear-theory relations, calculate the pressures in regions 2 through 5.
  - Calculate  $C_A$ ,  $C_N$ ,  $C_l$ ,  $C_d$ , and  $C_{m_{0.5c}}$  for this configuration.  $C_A$  is the axial force coefficient for the force along the axis (i.e., parallel to the chordline of the airfoil) and  $C_N$  is the normal force coefficient (i.e., normal to the chordline of the airfoil).



**Figure P10.10**

- 10.11.** Consider the two-dimensional airfoil having a section which is a biconvex profile, i.e.,

$$R(x) = 2t \frac{x}{L} \left( 1 - \frac{x}{L} \right)$$

The model is placed in a supersonic wind-tunnel, where the free-stream Mach number in the tunnel test section is 2.2;  $p_{t1} = 125$  lbf/in<sup>2</sup> and  $T_t = 600^\circ\text{R}$ . The airfoil is such that the maximum thickness ( $t$ ) equals  $0.07L$  and it occurs at the mid chord. The angle of attack is  $6^\circ$ .

Using the shock/Prandtl-Meyer expansion technique to model the flowfield for the biconvex airfoil section, calculate the pressure distributions both for the windward (bottom) and for the leeward (top) sides. Present the results in a single graph that compares the pressure distribution for the windward side with that for the leeward side.

- 10.12.** Consider the two-dimensional airfoil having a section which is a biconvex profile, i.e.,

$$R(x) = 2t \frac{x}{L} \left( 1 - \frac{x}{L} \right)$$

The model is placed in a supersonic wind-tunnel, where the free-stream Mach number in the tunnel test section is 2.2;  $p_{t_1} = 125 \text{ lbf/in}^2$  and  $T_t = 600^\circ\text{R}$ . The airfoil is such that the maximum thickness ( $t$ ) equals  $0.07L$  and it occurs at the mid chord. The angle of attack is  $6^\circ$ .

Using the linearized theory relations to model the flowfield for the biconvex airfoil section, calculate the pressure distributions both for the windward (bottom) and for the leeward (top) sides. Present the results in a single graph that compares the pressure distribution for the windward side with that for the leeward side.

- 10.13.** Consider the two-dimensional airfoil having a section which is a biconvex profile, i.e.,

$$R(x) = 2t \frac{x}{L} \left( 1 - \frac{x}{L} \right)$$

The model is placed in a supersonic wind-tunnel, where the free-stream Mach number in the tunnel test section is 2.2;  $p_{t_1} = 125 \text{ lbf/in}^2$  and  $T_t = 600^\circ\text{R}$ . The airfoil is such that the maximum thickness ( $t$ ) equals  $0.07L$  and it occurs at the mid chord.

Calculate  $C_d$  (the section drag coefficient) for the bi-convex airfoil section over an angle of attack range from  $0^\circ$  to  $10^\circ$ . Use the linearized theory relations to determine the pressure distribution that is required to calculate the form drag. Also, calculate  $C_d$  for the symmetric diamond-shaped airfoil section (see Problem 10.9).

Prepare a graph comparing the section drag coefficient ( $C_d$ ) for the bi-convex airfoil for the angle of attack range from  $0^\circ$  to  $10^\circ$  with that for the symmetric, diamond-shaped airfoil section. The shock/Prandtl-Meyer technique is to be used to calculate the pressure distribution for both airfoil sections. What is the effect of airfoil cross-section on the form drag?

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