Lecture 24

Prandtl-Glauert Rule

We shall now turn to ***subsonic*** flows past thin aerofoils. As follows from the analysis in Lecture 23, in order to describe the flow behaviour one has to solve the following equation for the potential perturbation function φ'

$$(1 - M_\infty^2)\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial y^2} = 0, \tag{24.1}$$

where M_∞ is the free stream Mach number.

The boundary conditions for equation (24.1) are the condition of attenuation of the perturbations far from the aerofoil

$$\varphi' \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \tag{24.2}$$

and the impermeability condition on the upper and lower aerofoil surfaces.

$$\frac{\partial \varphi'}{\partial y} = \begin{cases} V_\infty \dfrac{df_+}{dx} & \text{on} \quad y = 0+, \\ V_\infty \dfrac{df_-}{dx} & \text{on} \quad y = 0- \,. \end{cases} \tag{24.3}$$

Once a solution to the boundary-value problem (24.1)–(24.3) is found, one can calculate the pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = -\frac{2}{V_\infty}\frac{\partial \varphi'}{\partial x}. \tag{24.4}$$

Suppose now that the same aerofoil is placed in an incompressible fluid flow. In this case the velocity potential φ satisfies Laplace's equation (i.e. set $M_\infty = 0$ in (24.1) above)

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial x^2} = 0. \tag{24.5}$$

As the aerofoil is thin and induces only small perturbations in the flow field, we can write again

$$\varphi = V_\infty x + \tilde{\varphi}'(x, y). \tag{24.6}$$

Here $\tilde{\varphi}'$ is supposed small; the 'tilde' is used to distinguish the incompressible case from compressible one.

Substitution of (24.6) into (24.5) results in the following equation

$$\frac{\partial^2 \tilde{\varphi}'}{\partial x^2} + \frac{\partial^2 \tilde{\varphi}'}{\partial y^2} = 0. \tag{24.7}$$

In order to solve equation (24.7) we have to impose upon the potential perturbation function $\tilde{\varphi}'(x, y)$ the attenuation condition

$$\tilde{\varphi}' \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \tag{24.8}$$

and the impermeability condition on the aerofoil surface

$$\frac{\partial \tilde{\varphi}'}{\partial y} = \begin{cases} V_\infty \dfrac{df_+}{dx} & \text{on} \quad y = 0+, \\ V_\infty \dfrac{df_-}{dx} & \text{on} \quad y = 0- . \end{cases} \tag{24.9}$$

The pressure p may be found from Bernoulli's equation. For an incompressible flow we recall that it is written as

$$\frac{p}{\rho} + \frac{V^2}{2} = \frac{p_\infty}{\rho} + \frac{V_\infty^2}{2}.$$

Representing the velocity components in the form

$$u = V_\infty + \tilde{u}', \qquad v = \tilde{v}',$$

we can can easily show that Bernoulli's equation reduces to the linearized form

$$p = p_\infty - \rho V_\infty \tilde{u}'.$$

Further, taking into account that the perturbation of the longitudinal velocity is related to the perturbation of the potential as

$$\tilde{u}' = \frac{\partial \tilde{\varphi}'}{\partial x},$$

we see that the pressure coefficient

$$\tilde{C}_p = \frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = -\frac{2}{V_\infty}\tilde{u}' = -\frac{2}{V_\infty}\frac{\partial \tilde{\varphi}'}{\partial x}. \tag{24.10}$$

We shall now show that if the incompressible flow behaviour is known† then there is no need to solve the compressible problem (24.1)–(24.3). All the properties of the compressible flow may be easily found using the ***Prandtl-Glauert rule.*** It turns out that the compressible equations (24.1)–(24.4) may be converted into the incompressible form (24.7)–(24.10) using simple affine transformations. When making these transformations we do not want the independent variables x, y used in (24.7)–(24.10) to be confused with those in (24.1)–(24.4). For this reason we are going to denote x, y in (24.7)–(24.10) as $\tilde{x}$, $\tilde{y}$, respectively. Further, we allow the shape functions $\tilde{f}_\pm(\tilde{x})$ to be distinct from the compressible case. As a result the incompressible problem takes the form

$$\frac{\partial^2 \tilde{\varphi}'}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{\varphi}'}{\partial \tilde{y}^2} = 0. \tag{24.11}$$

$$\tilde{\varphi}' \to 0 \qquad \text{as} \quad \tilde{x}^2 + \tilde{y}^2 \to \infty, \tag{24.12}$$

$$\frac{\partial \tilde{\varphi}'}{\partial \tilde{y}} = \begin{cases} V_\infty \dfrac{d\tilde{f}_+}{d\tilde{x}} & \text{on} \quad \tilde{y} = 0+, \\ V_\infty \dfrac{d\tilde{f}_-}{d\tilde{x}} & \text{on} \quad \tilde{y} = 0-, \end{cases} \tag{24.13}$$

†From experimental observations or as a result of solving the boundary-value problem (24.7)–(24.9).

and the formula (24.4) for the pressure coefficient is rewritten as

$$\tilde{C}_p = -\frac{2}{V_\infty}\frac{\partial \tilde{\varphi}'}{\partial \tilde{x}}. \tag{24.14}$$

We now return to the compressible equations (24.1)–(24.4) and represent their solutions in the form

$$\varphi' = A\tilde{\varphi}', \qquad x = \tilde{x}, \qquad y = B\tilde{y}, \qquad f_\pm = C\tilde{f}_\pm, \tag{24.15}$$

where A, B and C are constant coefficients.

Substituting the expressions in (24.15) into (24.1)–(24.3) yields

$$(1 - M_\infty^2)A\frac{\partial^2 \tilde{\varphi}'}{\partial \tilde{x}^2} + \frac{A}{B^2}\frac{\partial^2 \tilde{\varphi}'}{\partial \tilde{y}^2} = 0, \tag{24.16}$$

$$\tilde{\varphi}' \to 0 \quad \text{as} \quad \tilde{x}^2 + \tilde{y}^2 \to \infty, \tag{24.17}$$

$$\frac{A}{B}\frac{\partial \tilde{\varphi}'}{\partial \tilde{y}} = \begin{cases} CV_\infty \dfrac{d\tilde{f}_+}{d\tilde{x}} & \text{on} \quad \tilde{y} = 0+, \\ CV_\infty \dfrac{d\tilde{f}_-}{d\tilde{x}} & \text{on} \quad \tilde{y} = 0-. \end{cases} \tag{24.18}$$

Comparing (24.16)–(24.18) with (24.11)–(24.13) it is easily seen, that setting

$$1 - M_\infty^2 = \frac{1}{B^2}, \qquad \frac{A}{B} = C \tag{24.19}$$

reduces the compressible equation to its incompressible form.

If we have identical aerofoils then $C = 1$ and, solving equations (24.19), we find

$$A = \frac{1}{\sqrt{1 - M_\infty^2}}, \qquad B = \frac{1}{\sqrt{1 - M_\infty^2}}.$$

Thus, the Prandtl-Glauert transformation (24.15) takes the form

$$\varphi' = \frac{1}{\sqrt{1 - M_\infty^2}}\,\tilde{\varphi}', \quad x = \tilde{x}, \quad y = \frac{1}{\sqrt{1 - M_\infty^2}}\,\tilde{y}, \tag{24.20}$$

and applying these to the pressure coefficient (24.4) gives

$$C_p = -\frac{2}{V_\infty}\frac{\partial \varphi'}{\partial x} = -\frac{2}{V_\infty}\frac{1}{\sqrt{1 - M_\infty^2}}\frac{\partial \tilde{\varphi}'}{\partial \tilde{x}} = \frac{\tilde{C}_p}{\sqrt{1 - M_\infty^2}}.$$

Thus, we can claim that ***the pressure coefficient*** C_p ***at any point on a thin aerofoil surface in an compressible flow is*** $(1 - M_\infty^2)^{-1/2}$ ***times the pressure coefficient*** $\tilde{C}_p$ ***at the same point on the same aerofoil in incompressible flow***. This is called the Prandtl-Glauert rule.

If we wish to have the same pressure coefficients over two different aerofoils, one in an incompressible flow and the other in a subsonic compressible flow, then we require $A = 1$ in (24.15) (i.e. $\phi' = \tilde{\phi}'$). We can achieve this by altering the aerofoil thickness; from (24.19) this yields

$$B = \frac{1}{\sqrt{1 - M_\infty^2}}, \qquad C = \sqrt{1 - M_\infty^2}.$$

Thus, the thickness of the aerofoil in the subsonic compressible flow is $\sqrt{1 - M_\infty^2}$ times the thickness of the incompressible aerofoil, i.e. $f_\pm = \sqrt{1 - M_\infty^2}\tilde{f}_\pm$.

Ackeret's Formula

If the flow past a thin aerofoil is ***supersonic*** ($M_\infty > 1$) then the general solution of equation (24.1) may be written as

$$\varphi'(x,y) = h(\xi) + g(\eta). \tag{24.21}$$

Here f and g are arbitrary functions of the arguments

$$\xi = x - \beta y, \qquad \eta = x + \beta y,$$

with constant β being related to the Mach number as $\beta = \sqrt{M_\infty^2 - 1}$. The straight lines $\xi =$ const. and $\eta =$ const. are called ***characteristics*** of the flow.

Let us restrict our attention to the flow above the aerofoil. It should be noted that unlike in subsonic flows, the perturbations produced in the supersonic flow below the aerofoil are incapable of penetrating into the flow region above the aerofoil and vice vérsa. Their mutual interference is confined to the region which lies downstream of the trailing edge between two characteristics emerging from the trailing edge. This means that the flow fields above and below the aerofoil are independent of one another and may be considered separately.

Notice that function $g(\eta)$ in (24.21) represents perturbations that propagate along the characteristics of the second family

$$x + \beta y = const.$$

These perturbations may exist in the flow field only if there is another object flying above and upstream of the aerofoil. Therefore, when dealing with an isolated aerofoil we have to set $g(\eta) = 0$ and write the solution of equation (24.1) as[‡]

$$\varphi'(x,y) = h(\xi), \qquad \xi = x - \beta y. \tag{24.22}$$

In order to determine function $h(\xi)$ we shall use the impermeability condition on the upper surface of the aerofoil, i.e. the top line of (24.3):

$$\left.\frac{\partial \varphi'}{\partial y}\right|_{y=0} = V_\infty \frac{df_+}{dx}. \tag{24.23}$$

Substitution of (24.22) into (24.23) leads to

$$-\beta h'(x) = V_\infty \frac{df_+}{dx}$$

which being integrated gives

$$h(x) = -\frac{V_\infty}{\beta} f_+(x).$$

So, we see that in the flow above the aerofoil the velocity potential perturbation function is

$$\varphi'(x,y) = -\frac{V_\infty}{\sqrt{M_\infty^2 - 1}} f_+(x - \beta y). \tag{24.24}$$

[‡]In the flow below the aerofoil one has to set $h(\xi) = 0$ so that

$$\varphi'(x,y) = g(\eta).$$

In order to find the pressure field we substitute (24.24) into (23.27), which yields

$$p = p_\infty + \rho_\infty V_\infty^2 \frac{f'_+(x-\beta y)}{\sqrt{M_\infty^2 - 1}}.$$

In particular, on the aerofoil surface, $y = 0$,

$$p = p_\infty + \rho_\infty V_\infty^2 \frac{f'_+(x)}{\sqrt{M_\infty^2 - 1}}. \tag{24.25}$$

The analysis may be repeated for a supersonic flow below the aerofoil, and it is easily found that the pressure field on the lower surface is given by

$$p = p_\infty - \rho_\infty V_\infty^2 \frac{f'_-(x)}{\sqrt{M_\infty^2 - 1}}. \tag{24.26}$$

Thus, taking into account that $f'_\pm(x)$ represents the angle $\theta_\pm(x)$ made by the tangent to the aerofoil contour (on the upper/lower side respectively) with the direction of the oncoming flow, we can write (24.25) and (24.26) as

$$p = p_\infty \pm \rho_\infty V_\infty^2 \frac{\theta_\pm(x)}{\sqrt{M_\infty^2 - 1}}. \tag{24.27}$$

The expressions (24.25), (24.26) or equivalently (24.27) are called ***Ackeret's formula***. This formula gives a convenient means to determine the Drag and Lift on a thin aerofoil in a supersonic flow.

Drag on a Supersonic Aerofoil

To illustrate how to employ Ackeret's formula we work through the example given in Exercise Sheet 9, Question 3. This requires us to determine the drag on a thin aerofoil in supersonic flow (shown in figure 24.1); the lift could be calculated in a similar fashion.

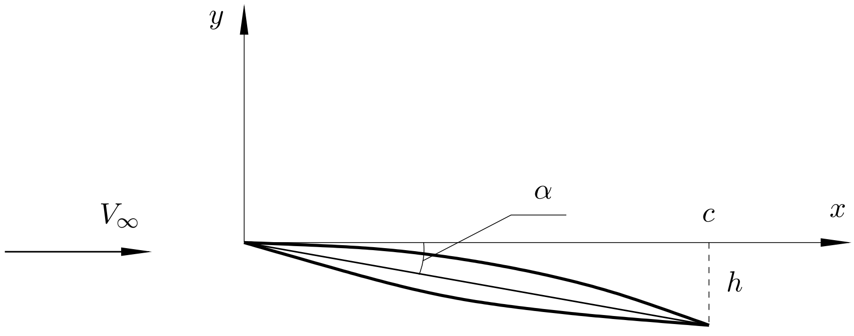

Figure 24.1: An aerofoil at angle of attack α in a supersonic free-stream.

We start with the upper surface of the aerofoil. The pressure on this surface may be calculated (see Figure 24.2) according to the Ackeret formula (24.25). The pressure force acting on a small element dl of the aerofoil contour is

$$P_+ = p_+ dl \approx p_+ dx. \tag{24.28}$$

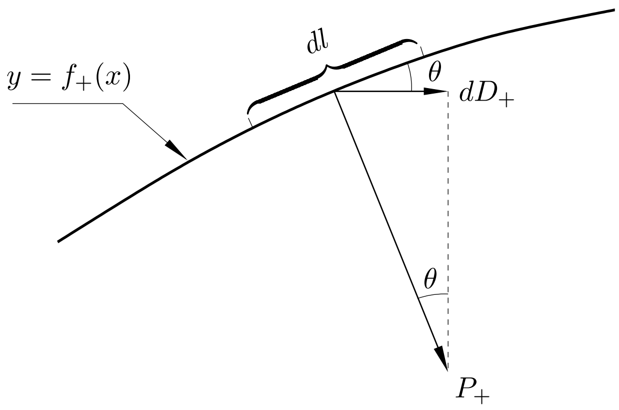

Figure 24.2: The upper surface of the aerofoil.

The approximaton takes account of the fact that the aerofoil is thin, and therefore $dl \approx dx$. The x-component of this force represents a contribution of the surface element dl to the aerofoil drag

$$dD_+ = P_+ \sin\theta \approx P_+\theta = P_+ \frac{df_+}{dx}. \tag{24.29}$$

Substitution of (24.25) and (24.28) into (24.29) yields

$$dD_+ = \left(p_\infty + \rho_\infty V_\infty^2 \frac{df_+/dx}{\sqrt{M_\infty^2 - 1}}\right) \frac{df_+}{dx} dx,$$

which, being integrated over the entire upper surface, gives

$$D_+ = \int_0^c \left(p_\infty + \rho_\infty V_\infty^2 \frac{df_+/dx}{\sqrt{M_\infty^2 - 1}}\right) \frac{df_+}{dx} dx.$$

Now we turn to the lower surface of the aerofoil (see Figure 24.3). The pressure on this surface is given by Ackeret's formula in the form (24.26). The pressure force acting

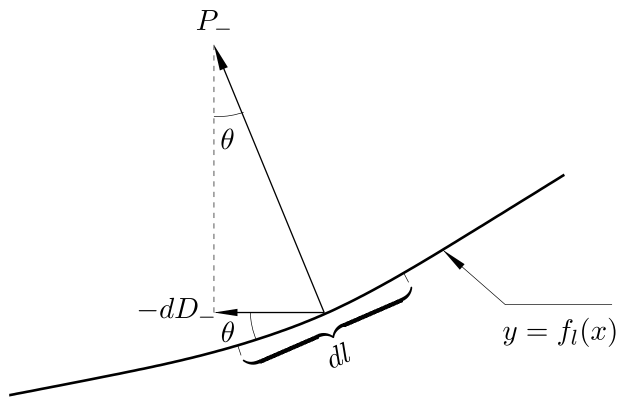

Figure 24.3: The lower surface of the aerofoil.

on a small element dl of this surface is

$$P_- = p_- dl,$$

and so gives a contribution to the aerofoil drag

$$dD_- = -P_- \sin\theta = -p_- dl \sin\theta.$$

As we are dealing with a thin aerofoil, we can write

$$dD_- \approx -p_- dx\,\theta = -p_- \frac{df_-}{dx} dx. \tag{24.30}$$

Substituting (24.26) into (24.30) and integrating over the entire lower surface of the aerofoil gives

$$D_- = \int_0^c \left(-p_\infty + \rho_\infty V_\infty^2 \frac{df_-/dx}{\sqrt{M_\infty^2 - 1}} \right) \frac{df_-}{dx} dx.$$

Hence, the total drag of the aerofoil is

$$\begin{aligned} D = D_+ + D_- = p_\infty \int_0^c \left(\frac{df_+}{dx} - \frac{df_-}{dx} \right) dx \\ + \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^c \left[\left(\frac{df_+}{dx} \right)^2 + \left(\frac{df_-}{dx} \right)^2 \right] dx. \end{aligned} \tag{24.31}$$

Since

$$f_+(0) = f_-(0), \qquad f_+(c) = f_-(c),$$

the first integral in (24.31) vanishes, and we can confirm that

$$D = \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^c \left[\left(\frac{df_+}{dx} \right)^2 + \left(\frac{df_-}{dx} \right)^2 \right] dx. \tag{24.32}$$

We can simplify the expression for the drag by measuring the angle of the upper and lower surfaces about the chord angle α (see figure 24.1), i.e.

$$\frac{df_+}{dx} = -\alpha + \theta_+(x), \qquad \frac{df_-}{dx} = -\alpha + \theta_-(x).$$

This allows us to express formula (24.32) in the form

$$D = \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^c \left\{ 2\alpha^2 - 2\alpha\theta_+(x) - 2\alpha\theta_-(x) + \big[\theta_+(x)\big]^2 + \big[\theta_-(x)\big]^2 \right\} dx,$$

and since

$$\int_0^c \theta_+(x) dx = 0, \qquad \int_0^c \theta_-(x) dx = 0,$$

we arrive at the conclusion that

$$D = \frac{\rho_\infty V_\infty^2}{\sqrt{M_\infty^2 - 1}} \int_0^c \left\{ 2\alpha^2 + \big[\theta_+(x)\big]^2 + \big[\theta_-(x)\big]^2 \right\} dx.$$

Finally, this may be expressed as the drag coefficient:

$$C_D = \frac{2}{c\sqrt{M_\infty^2 - 1}} \int_0^c \left\{ 2\alpha^2 + \big[\theta_+(x)\big]^2 + \big[\theta_-(x)\big]^2 \right\} dx. \tag{24.33}$$