Momentum (Newton’s 2\textsuperscript{nd} Law of Motion)

Case 3 – Airfoil Drag

A very important application of Momentum in aerodynamics and hydrodynamics is the calculation of the drag of a body by calculating the momentum loss in its wake profile.

Case 3.1 – Airfoil in free flight

![Wake profiles of an airfoil. The profiles are made visible in water flow by pulsing a voltage through a straight wire perpendicular to the flow, creating small bubbles of hydrogen that subsequently move downstream with the flow. [Yasuki Nakayama, Tokai University, Japan – taken from Anderson’s Fundamentals of Aerodynamics book]](image-url)

When fluid moves past a body the viscosity of the fluid causes it to stick to the surface of the body (no-slip condition). This creates very large velocity gradients normal to the surface of the body. These velocity gradients give rise to viscous shear stresses according to Newton’s Law of viscosity:

$$\tau = \mu \frac{du}{dy}$$

These shear stresses, integrated over the surface of the body result in skin friction drag. The sum of skin friction drag and pressure (form) drag, which results from flow separation on the body, is known as the profile drag of the body.

From Newton’s 3\textsuperscript{rd} Law, the body then exerts a force of equal magnitude but in opposite direction on the fluid, reducing its momentum. Hence, when a fluid moves past a body
we observe that the momentum of the fluid directly behind the body is significantly reduced, as shown in Figure 1.

**Case 3.1.1 – Top and bottom control surfaces chosen to be streamsurfaces**

*Figure 2 – Schematic of the control volume around an airfoil in free flight. The airfoil is excluded from the control volume.*

**Assumptions:**
- Steady flow
- Incompressible flow
- 2-D flow
- Viscous effects along the control surface are negligible

*What else is going on:*
- One flow stream
- Non-uniform velocity profile downstream
- Flow speed changes
- Flow cross-sectional area changes

The selection of the control volume is key in simplifying the calculation.
The airfoil is excluded from the control volume, so that the forces $\vec{F}_H$ and $\vec{F}_V$ from the airfoil on the air (Figure 3) will show up as external forces on the control volume. From Newton’s 3rd Law:

$$\vec{F}_H = -\vec{D}$$
$$\vec{F}_V = -\vec{L}$$

where $\vec{D}$ and $\vec{L}$ are the drag and lift forces of the airfoil. If the airfoil were included in the control volume, these four forces would all be internal to the control volume and cancel each other out. As a result, none of these forces would appear in Momentum.

Streamsurfaces are used for the upper and lower bounds of the control volume, so there is no flow crossing the top and bottom control surface.

The control surface is taken far enough from the body in all directions, so that the pressure on it may be assumed uniform, equal to ambient atmospheric pressure:

$$p = p_\infty$$

The width of the control volume into the paper may be taken as $b$, the wingspan or simply as one unit of length.

**Continuity**

$$\dot{m}_1 = \dot{m}_2$$

$$\dot{m}_1 = \rho u_1 A_1 = \rho u_1 (h_1 b)$$

$A_2 \gg A_1$ to maintain the same mass flow rate because the fluid slows down behind the airfoil. Also, since the velocity profile at 2 is non-uniform, an integral is needed to describe the mass flow rate there:

$$\dot{m}_2 = \rho b \int \limits_2 u_2(y) dy$$
**Momentum**

Figure 3 – Freebody diagram for the control volume surrounding the airfoil: $p_\infty$ indicates the local pressure along the control surface, while the arrows around the control surface indicate local pressure forces.

The airfoil exerts a viscous force (through its boundary layer) on the fluid flowing past and it slows it down, so:

$$\dot{M}_{x,\text{out}} < \dot{M}_{x,\text{in}}$$

$x-$Momentum: \[ F_H = \dot{M}_{x,\text{out}} - \dot{M}_{x,\text{in}} \]

The velocity profile at 1 is uniform, so we can write the momentum there using an algebraic expression:

$$\dot{M}_{x,\text{in}} = \rho u_1^2 (bh_1)$$

The velocity profile at 2 is non-uniform and we need an integral to describe the momentum there:

$$\dot{M}_{x,\text{out}} = \rho b \int_{y_2}^{y_1} u_2^2(y) dy$$

$x-$Momentum: \[ -F_H = D = \rho b \int_{y_2}^{y_1} u_2^2(y) dy - \rho bh u_1^2 \]
Case 3.1.2 – Top and bottom control surfaces chosen parallel to the x-axis

Assumptions:

- Steady flow
- Incompressible flow
- 2-D flow
- Viscous effects along the control surface are negligible

What else is going on:

- One flow stream
- Non-uniform velocity profile downstream
- Flow speed changes
- Constant flow cross-sectional area: $A_1 = A_2$

This control volume appears simpler in shape than the one chosen before. However, it is now more complex to write Continuity and Momentum. The top and bottom control surface are not streamsurfaces. This implies that there is mass and momentum leakage from both of these surfaces, so $\dot{m}_2 < \dot{m}_1$.

**Continuity**

$A_2 = A_1 \rightarrow$ for the flow to be steady, i.e. no mass accumulation inside the control volume:

$$\dot{m}_1 = \dot{m}_{\text{top}} + \dot{m}_{\text{bottom}} + \dot{m}_2$$
as before \[ \dot{m}_1 = \rho u_1 A_1 = \rho u_1 (h_1 b) \]
and \[ \dot{m}_2 = \rho b \int u_2 (y) dy \]
so \[ \dot{m}_{\text{top}} + \dot{m}_{\text{bottom}} = \rho u_1 (h_1 b) - \rho b \int u_2 (y) dy \]
so \[ \dot{m}_{\text{top}} = \dot{m}_{\text{bottom}} = \frac{1}{2} \left[ \rho u_1 (h_1 b) - \rho b \int u_2 (y) dy \right] \]

\textit{Momentum}

Figure 5 – Freebody diagram for the control volume surrounding the airfoil: \( p_\infty \) indicates the local pressure along the control surface, while the arrows around the control surface indicate local pressure forces.

\( x \)-Momentum: \[ F_H = \dot{M}_{x,\text{out}} - \dot{M}_{x,\text{in}} = \dot{M}_{x,\text{out,top+bottom}} - \dot{M}_{x,\text{in}} \]
\[ \dot{M}_{x,\text{in}} = \rho u_1^2 (bh_1) \]
\[ \dot{M}_{x,\text{out}} = \rho b \int u_2^2 (y) dy \]

It is assumed that the flow velocity through the top and bottom control surface is only slightly inclined with respect to the \( x \)-axis, hence
\[ u_{\text{top}} = u_{\text{bottom}} = u_1 \]
\[ \dot{M}_{x,\text{out,top+bottom}} = 2 \dot{m}_{\text{top}} u_1 \]
\[ x - \text{Momentum: } -F_H = D = \rho b \int_2^1 u_2^2(y) \, dy + 2\dot{m}_{\text{top}} u_1 - \rho bh_1 u_1^2 \]

where \( \dot{m}_{\text{top}} \) was calculated above from Continuity.
Case 3.1.3 – Non-uniform pressure in the wake

Figure 6 – Freebody diagram for the control volume surrounding the airfoil. Note that \( p_\infty \) and \( p_2(y) \) indicate the local pressure along the control surface, while the arrows around the control surface indicate local pressure forces.

If the control surface at the exit of the control volume is close to the airfoil the pressure at 2 will be affected by the presence of the airfoil and \( p_1 \neq p_2 \). Now in addition to the viscous force from the airfoil there is also a pressure force on the control volume:

\[
F_H + \left[ p_1 A_1 + p_1 (A_2 - A_1) - b \int p_2(y) dy \right] = \rho b \int \frac{u_2^2}{2} dy - \rho bh_1 u_1^2
\]

It has been assumed that the pressure on the top and bottom portions of the control surface has remained equal to atmospheric pressure. The pressure forces on each of these surfaces (top + bottom) act normal to the surface. If we take the \( x \) – component of these pressure forces at each point of the top and bottom surfaces we get \( p_1 (A_2 - A_1) \), which is precisely the 2\(^{\text{nd}}\) term in the bracket in the \( x \) – Momentum. In this case the drag is calculated from:

\[
-F_H = D = \rho b \int \frac{u_2^2}{2} dy - \rho bh_1 u_1^2 + b \left[ p_1 h_2 - \frac{1}{2} \int p_2(y) dy \right]
\]
Case 3.2 – Airfoil in a wind tunnel test section

The easiest way to measure airfoil drag is in a wind tunnel. To avoid 3-D (tip) effects airfoils are mounted from wall to wall in the test section of a wind tunnel.

![Airfoil in a wind tunnel test section](image)

**Figure 7 – Schematic of an airfoil in a wind tunnel test section.**

**Assumptions:**

- Steady flow
- Incompressible flow
- 2-D flow
- Viscous effects along the wind tunnel walls are neglected

**What else is going on:**

- One flow stream
- Non-uniform velocity profile downstream
- Flow speed changes from streamline to streamline
- Flow cross-sectional area remains constant: $A_1 = A_2$

Since the airfoil is inside a wind tunnel the top and bottom control surface are by default the wind tunnel walls. The flow cannot cross these walls, hence they must be streamsurfaces of the flow. In a real wind tunnel flow, the no slip condition requires zero velocity at each wall. This condition, however, does not contradict having the top and bottom walls as streamsurfaces of the flow.

**Continuity**

\[
\dot{m}_m = \dot{m}_{out} \\
\rho u_1 A = \rho \int_{2} u_2(y) dy = \rho \bar{u}_2 A
\]
Hence, no matter what the wake profile looks like in section 2, Continuity requires that the average velocity at 2 is the same as the average velocity at 1.

\[ u_1 = \bar{u}_2 \quad \text{but} \quad \bar{u}_2 = \frac{1}{h} \int u_2(y) \, dy \quad \text{and} \quad \bar{u}_2^2 = \frac{1}{h} \int u_2^2(y) \, dy \]

so \[ \bar{u}_2^2 \neq u_2^2 \Rightarrow \dot{M}_{x,1} = \rho u_1^2 A \neq \rho u_2^2 A = \dot{M}_{x,2} \]

**Momentum**

*Figure 8 – Freebody diagram for the control volume surrounding an airfoil in a wind tunnel test section: \( p_1 \), \( p_u(x) \), \( p_r(x) \), \( p_2 \) indicate local pressure along the control surface, while the arrows around the control surface indicate local pressure forces.*

\( x - \text{Momentum:} \)

\[
-F_H = D = \rho b \int_0^2 u_2^2(y) \, dy - \rho bh u_1^2 + (p_2 - p_1) bh
\]

The pressure in the downstream section has been assumed constant across the height of the test section but not the same as the pressure upstream. Space constraints in the wind tunnel dictate measuring the wake traverse fairly close to the airfoil and this results in \( p_2 \neq p_1 \).

The pressure on the top and bottom wind tunnel walls (as well as along the top and bottom control surface) cannot be the same if the airfoil is lifting. When an airfoil lifts it pushes the air down, hence, we expect the pressure on the lower wall to be higher than the pressure on the upper wall. These pressure forces must be included in the \( y - \text{Momentum eq.} \)
y – Momentum:

\[ F_y + b \left( \int_{\text{Lower CS}} p_l(x) \, dx - \int_{\text{Upper CS}} p_u(x) \, dx \right) = \dot{M}_{y,\text{out}} - \dot{M}_{y,\text{in}} = 0 \]

Obviously, there is no vertical momentum crossing the control surface at any point, so our control surface is in equilibrium in the y – direction, as shown by the y – Momentum. Hence the lift of the airfoil is:

\[ -F_y = L = b \left( \int_{\text{Lower CS}} p_l(x) \, dx - \int_{\text{Upper CS}} p_u(x) \, dx \right) \]

can be found by integrating and subtracting the p – distributions on the bottom and top wind tunnel walls.