Viscosity effects on weak irregular reflection of shock waves in steady flow

Mikhail S. Ivanov, Yevgeniy A. Bondar, Dmitry V. Khotyanovsky*, Alexey N. Kudryavtsev, Georgy V. Shoev

Computational Aerodynamics Laboratory, Khrustianovich Institute of Theoretical and Applied Mechanics (ITAM), Siberian Branch of the Russian Academy of Sciences, Institutskaya 4/1, Novosibirsk 630090, Russia

A B S T R A C T

The viscosity effects on weak shock wave reflection are investigated with the Navier–Stokes and DSMC flow solvers. It is shown that the viscosity plays a crucial role in the vicinity of three-shock intersection. Instead of a singular triple point, in viscous flow there is a smooth three shock transition zone, where one-dimensional shock jump relations cannot be applied. At the flow parameters corresponding to the von Neumann reflection, when no inviscid three-shock solution exists, the computations predict an irregular shock-wave configuration similar to that observed previously in experiments. The existence of a viscous zone in the region of shock-wave interaction allows a continuous transition from the parameters behind the Mach stem to the parameters behind the reflected shock, which is impossible in the inviscid three-shock theory. Results of numerical simulations agree qualitatively with the conclusions of the approximate viscous theory, which describes the flow in the vicinity of the three-shock intersection.

© 2009 Elsevier Ltd. All rights reserved.

Contents

1. Background and motivation ................................................................. 89
2. Problem formulation and flow conditions ............................................. 92
3. Numerical methods ............................................................................ 93
4. Weak irregular reflection for different wedge angles .............................. 96
5. Comparison of computational results with the Sternberg model .......... 100
6. Conclusion ....................................................................................... 101

Acknowledgments .................................................................................. 103
Appendix I. Transfer of numerical data in the planes (U, V), (θ, p), (θ, ρ), and (θ, T) .............. 104
Appendix II. Numerical accuracy ............................................................... 104
References ............................................................................................. 104

1. Background and motivation

Many interesting phenomena that occur in oblique shock wave reflection have been discovered in the past. The main feature herein is the existence of two possible configurations of shocks, regular and irregular. Regular reflection consists of an incident shock wave and a reflected shock wave with supersonic flow behind the reflected shock. Irregular reflection, which is in most cases called Mach reflection after E. Mach who first discovered this phenomenon, is a complex shock wave pattern that combines incident and reflected shock waves and a Mach stem. A contact discontinuity (slip surface) emanates from the triple point due to inequality of entropy in the flow passing through the incident and reflected shocks and the flow passing through the Mach stem. Classical theoretical methods such as shock polar analysis and the three-shock theory based on the Rankine–Hugoniot jump conditions across the oblique shocks were developed by von Neumann to describe the shock wave configurations at various flow parameters and to predict transitions between different types of shock wave interaction. These theoretical methods were mainly developed for pseudo-steady flow, i.e. for diffraction of a plane...
incident shock on a rigid ramp of an infinite span. Later, the theoretical models were applied to the interaction of standing incident shock waves generated by symmetrical wedges in a steady supersonic flow (see, e.g. [1]). These theoretical methods predict well most of the features of shock wave interaction. Recent success of the von Neumann’s theory is the discovery of the hysteresis in the regular–Mach reflection transition for strong shock waves in steady flows (see review in [2] and the experimental results of [3]). The hysteresis phenomenon in steady reflection of strong shock waves was predicted by Hornung [4] on the basis of two transition criteria formulated by von Neumann [5], viz. the detachment criterion and the mechanical equilibrium criterion (von Neumann criterion).

Steady shock wave reflection is very important in aerodynamics and has been extensively studied in recent years with an emphasis on strong shock waves (for flow Mach numbers higher than 2.2 in air) [2]. For supersonic civil aviation, however, the lower Mach number range is of greatest interest. Regular and irregular interactions of different types are inherent in such critical phenomena as off-design inlet flows, inlet starting, and flow stalling. Interactions and reflections of weak shock waves are typical for supersonic inlet flows at low and moderate Mach numbers ($M=1–2$). There are many problems of irregular shock reflections in steady flows, which are not yet investigated. One of the most exciting phenomena that occur in irregular reflection of weak shock waves is a shock wave reflection in the range of flow parameters where the von Neumann’s three-shock theory does not produce any solution whereas the experiments reveal a shock wave configuration similar to the Mach reflection pattern. This inconsistency was first discovered in shock-tube experiments [6,7] and was later confirmed in numerous studies on pseudo-steady shock wave reflection [8–12]. It is referred to as the von Neumann paradox, and the observed irregular reflection pattern is called the von Neumann reflection (vNR). Guderley [13,14] developed a first non-contradictory theoretical model for weak shock wave reflection that assumed the existence of a Prandtl–Meyer expansion and a local supersonic patch behind the triple point.

There are roughly two main ways to overcome the deficiency of the three-shock theory in description of the irregular reflection of weak shock waves. In the first group of studies, the perfect-fluid assumption is kept. As soon as the inviscid three-shock theory fails to predict solution at the von Neumann paradox conditions, some “non-traditional” gasdynamics models were developed, e.g. [8,15–17] sacrificed the natural requirement of the three-shock theory that the flow passing through the reflected shock is parallel to the flow passing through the Mach stem. First effort of numerical simulation of the shock wave reflection at the conditions of the von Neumann paradox was made by Colella and Henderson [10] with the Euler equations and a second-order shock-capturing scheme with adaptive mesh refinement. Based on their results Colella and Henderson [10] proposed a hypothesis that the reflected shock near the triple point degenerates into a continuous compression wave. There is however some controversy concerning their results because of the influence of numerical viscosity and grid resolution in their computations. It is well known that the discretization of the Euler equations with shock-capturing schemes always includes some numerical

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Parameters for different wedge angles, $M=1.7$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_w$ (deg)</td>
<td>$p_w/p_s$</td>
</tr>
<tr>
<td>Case 1</td>
<td>12</td>
</tr>
<tr>
<td>Case 2</td>
<td>9.627</td>
</tr>
<tr>
<td>Case 3</td>
<td>13</td>
</tr>
<tr>
<td>Case 4 (vNR)</td>
<td>13.5</td>
</tr>
</tbody>
</table>
dissipation inherent in the algorithm, so the details of the scheme are very important in this problem. The computations of [18,19] performed with ultra-high resolution using a novel shock-fitting technique confirmed the basic predictions of the Guderley’s theory. An expansion fan emanating from the triple point and the adjacent supersonic patch were observed behind the reflected shock wave in computations of [18,20]. Based on these results a new four-wave model was formulated, which eliminates most difficulties in theoretical treatment of the shock reflection at the von Neumann paradox conditions [21]. Recent numerical investigations of the von Neumann paradox with the use of the Euler equations [22] showed the existence of a more complex flowfield with multiple supersonic patches. The above mentioned multiple supersonic patches were also detected in a recently performed experimental study [23]. However, it is too early to speak of unambiguous experimental confirmation of their existence because numerical and experimental sizes of the supersonic patches are more than an order different (less than 1% of the Mach stem height in the computations and about 10% in the experiment).

All these studies were made for pseudo-steady shock wave reflection. In steady flows, there were very few attempts of inviscid numerical simulations of the shock wave reflection at the conditions of the von Neumann paradox (see recent papers [24,25]).

In the second group of works, the vNR is explained by the effects of flow viscosity [26,27]. Owing to a finite thickness of the interacting shock waves, the flow in the vicinity of the triple point is substantially different from the inviscid flow. As stated by Sternberg in [26], at the intersection there must be a zone of essentially two-dimensional flow, which was labeled a “non Rankine–Hugoniot shock wave zone”, where the gradients of flow parameters in the direction tangent to the shock wave are important. According to this Sternberg model the entire flow domain is divided into three sub-domains: upper and lower sub-domains where the Rankine–Hugoniot relations hold and the transition sub-domain where these relations do not apply. This non Rankine–Hugoniot zone acts as a buffer zone separating Rankine–Hugoniot shocks in the upper and lower sub-domains.

It should be noted that recent numerical research [28] based on the Navier–Stokes and Boltzmann equations demonstrated the presence of a non-Rankine–Hugoniot zone in the case of Mach reflection of strong shock waves at Mach number $M=4$. In particular, this zone is characterized by a 10% increase in pressure compared to the inviscid three shock solution. The linear size of this region is several tens of local mean free paths. The structure of this zone is almost unchanged within the range of Reynolds numbers $Re$ from 5000 to 10000 in spatial coordinates normalized to the free-stream mean free path.

The basic goal of the present study was a numerical investigation of viscous effects in irregular reflection of weak shock waves. To confirm the validity of numerical results, the computations were performed by two principally different approaches: solving the Navier–Stokes equations and solving the Boltzmann equation by direct simulation Monte Carlo method (DSMC). The first stage of research includes viscous computations of the flow under conditions where the three-shock theory predicts the existence of the Mach reflection. Studying the viscous effects for such conditions is a necessary step before the second stage of research, which is also described in the present paper, namely viscous calculations under the conditions of the von Neumann reflection. The present paper also describes the computations of the von Neumann reflection for conditions of [26] and the numerical results are compared with the estimates based on the approximate theory [26].

Fig. 3. Schematic of the Navier–Stokes and DSMC computational domains and boundary conditions. IS—incident shock, RS—reflected shock, MS—Mach shock, EF—expansion fan.

Fig. 4. Ratio of the local mean free path to the grid cell size $M=1.7$, $Re=2122.9$, $Kn=0.001$, $\gamma=5/3$, and $\theta_0=9.63^\circ$. 

$\lambda/\Delta x$

<table>
<thead>
<tr>
<th>\lambda/\Delta x</th>
<th>1.30</th>
<th>1.55</th>
<th>1.68</th>
<th>1.80</th>
<th>1.93</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda/\Delta x$</td>
<td>2.30</td>
<td>2.18</td>
<td>2.05</td>
<td>1.93</td>
<td>1.80</td>
</tr>
</tbody>
</table>
2. Problem formulation and flow conditions

In this study we investigate irregular reflection of weak shock waves in a steady supersonic flow with a Mach number $M$ between two symmetrical wedges with identical angles $\theta_w$. The flow pattern for the Mach reflection is shown in Fig. 1. The reflection of the incident shock (IS) occurs on the plane of symmetry half-way between the wedges. A reflected shock wave, RS, and a Mach stem, MS, are formed at the intersection triple point T. In addition to three shock waves, a slip surface, SS, also emanates from the triple point T and separates the streams that pass through the MS and through two oblique shocks, IS and RS. The trailing edge of the wedge generates an expansion fan that is refracted on the RS and then interacts with the SS. Owing to this...
interaction the slip surface becomes curved and forms a "virtual nozzle". The cross-section of the stream tube between two symmetrical slipstreams decreases initially to a minimum at the sonic throat and then increases again. As a result, initially subsonic flow behind the Mach stem accelerates up to supersonic velocity. The Mach stem cross-sectional area is related to the area of the sonic throat of the virtual nozzle. The size and position of the sonic throat, and consequently the Mach stem, is controlled by a geometrical parameter $g/w$, which is the ratio of the distance between the wedges $g$ to the wedge chord $w$.

A characteristic feature of the weak shock wave reflection is that the flow behind the reflected shock wave of the Mach reflection is subsonic. For a gas with the ratio of specific heats $\gamma$ there is a Mach number value $M_\gamma$ at which the shock wave angles corresponding to the detachment and the von Neumann [see (5)] criteria coincide: $M_\gamma \approx 2.2$ for diatomic gases with $\gamma=1.4$, and $M_\gamma \approx 2.47$ for monatomic gases with $\gamma=5/3$. In the present study the shock-wave reflections with flow Mach numbers $M \leq M_\gamma$ are considered. Case distinction between weak and strong shock wave reflection is discussed in [1].

It is convenient to illustrate various types of shock interactions on pressure-deflection diagrams, see Fig. 2, where different shock polar combinations for $\gamma=5/3$ and $M=1.7$ are given. The intersection of the reflected shock polar with the incident shock polar indicates matching pressure $p_{rs}$ and flow deflection at the slip surface $\theta_{rs}$ and represents a three-shock solution. These solutions are listed in Table 1 for various wedge angles $\theta_w$. For reference, the Mach number behind the reflected shock $M_\gamma$ is also given in the last column of Table 1. At $\theta_w=8.5^\circ$, the shock polar intersection yields a three-shock solution corresponding to an MR configuration with a subsonic flow behind the reflected shock wave. The deflection angle of the flow passing through the reflected shock decreases. At $\theta_w=9.627^\circ$, the reflected shock wave becomes normal to the flow behind IS, and the flow deflection is not changed. At higher wedge angles, the reflected shock wave predicted by the three-shock solution is inclined forward with respect to the flow behind IS, and the flow deflection in the reflected shock increases. The three-shock solution obtained at $\theta_w=13^\circ$ predicts an incoming reflected shock wave with a supersonic flow behind it. Such a configuration is unphysical unless it is maintained by the imposed upstream boundary conditions, which is apparently not the case here. In our problem, such a shock wave should have appeared from "nowhere". Since small disturbances cannot propagate upstream in a supersonic flow behind the reflected shock wave, there is no reason for such a reflected shock to exist, and this case is discarded. The angle $\theta_w=13.5^\circ$ does not produce any three-shock solution since the reflected shock polar does not intersect the incident shock polar. This last case corresponds to the von Neumann paradox conditions. The vNR domain lies in the range of $\theta_w$ between $\theta_w=13.38^\circ$, i.e., the highest deflection angle at which the three-shock solution still exists, and $\theta_w=14.16^\circ$, which corresponds to sonic conditions behind the incident shock.

In this study, we investigate numerically some of these shock wave reflection configurations for Mach number $M=1.7$ (see conditions for Cases 1–4 in Table 1).

3. Numerical methods

The computations are conducted with the Navier–Stokes (NS) equations and the direct simulation Monte Carlo (DSMC) method [29]. The Navier–Stokes code [30] is a time-explicit shock-capturing code based on 5th order WENO reconstruction [31] of convective fluxes and central 4th order approximation of dissipation terms. The NS computations were run in a rectangular computational domain (see Fig. 3) with uniform grid spacing. The left boundary of the computational domain is a supersonic inflow with the free-stream flow parameters imposed. The right boundary of the domain was placed far enough downstream to

Fig. 8. Density fields for different wedge angles. DSMC computations, $M=1.7$, $\gamma=5/3$, and Re=2122.9 (Kn=0.001). The black bold lines show the shock-wave positions predicted by the three-shock theory. Case 1, wedge angle $\theta_w=9.63^\circ$ (a); Case 2, wedge angle $\theta_w=12^\circ$ (b); Case 3, wedge angle $\theta_w=13^\circ$ (c); Case 4, von Neumann paradox conditions, wedge angle $\theta_w=13.5^\circ$ (d).
ensure supersonic outflow conditions there. Extrapolation of the flow variables was used to impose boundary conditions at the outflow. At the lower boundary of the domain, symmetrical boundary conditions (mirror reflection) were used. The upper boundary was placed at \( y = g \) corresponding to the vertical position of the trailing edge of the wedge. The boundary conditions on the upper boundary were specified using a special procedure to maintain flow conditions at the horizontal line \( y = g \). Supersonic free-stream conditions were specified along the segments 1–2 of the upper boundary (see Fig. 3). The segments 2–3 correspond to the intersection of the upper boundary with the incident shock, where in a viscous case a smooth variation of the flow parameters inside the shock wave should be specified. For the internal structure of the shock wave we used the analytical solution of the one-dimensional Navier–Stokes equations that can be found in [32]. Along the line segments 3–4 the flow parameters corresponding to Rankine–Hugoniot conditions behind the incident shock were imposed. Along the segment 4–5 the inviscid wall (symmetry) boundary conditions were used. The computations were started with a uniform supersonic flow filling the entire computational domain. The numerical solution was then advanced in time with the 2nd order Runge–Kutta scheme until a steady state was achieved. Convergence to the steady state was established when the variation of the right hand sides of the Navier–Stokes equations at a given time step in \( L_\infty \) and \( L_1 \) norms dropped below a certain small value. Additionally, the characteristic flow features, such as shock positions, were also monitored during the computations to ensure their time independence.

The DSMC method is a numerical technique where the gas flow is presented as an ensemble of model particles, each representing a large number \((\sim 10^{12}–10^{20})\) of real molecules. The modeling process is divided into two independent stages: collisional relaxation and free-molecular transfer with a time step \( \Delta t \). At the first stage, molecular collisions are simulated in each “collisional” cell, disregarding the mutual positions of the molecules. Then, at each time step \( \Delta t \), the molecules in all cells are shifted by distances proportional to their velocities. If the molecule collides with the body surface during its free-molecular travel, molecule reflection is modeled in accordance with a specified law of gas-surface interaction. The spatial distributions of gas-dynamic parameters, such as velocity, density, temperature, etc., are obtained by averaging molecule properties sampled in each cell over some time interval after reaching the steady state. The DSMC method is actually a method of the numerical solution of the kinetic Boltzmann equation [33].

The DSMC computations were performed by the statistical modeling in low-density environment (SMILE) software system [33]. The DSMC computational domain is shown in Fig. 3; its boundaries are plotted by bold solid lines. Separate rectangular grids were used for modeling molecular collisions and sampling the gas dynamic parameters. The first grid was uniform and linear.
cell size $\Delta x$ was less than the minimum value of the mean free path in the computational domain (see Fig. 4, where the ratio of the mean free path to the collisional cell size is shown for the typical computation). The second grid was condensed in the zones of interest: the Mach stem and the point of the shock reflection from the symmetry plane for Mach and regular reflection, respectively. At the initial moment, the domain was populated by the model particles according to the Maxwell distribution function corresponding to the free-stream parameters. Free-stream conditions were imposed on the left boundary and on a portion of upper boundary of the computational domain. The right (downstream) boundary was selected so that the flow there was supersonic. Specular reflection condition was used at the lower boundary (the symmetry plane), the wedge surface (to avoid the viscosity effects on the wall) and the part of the upper boundary ($y=g$).

The computations are performed for a low-Reynolds-number flow (typically, $Re \sim 1000–5000$) with full resolution of the internal structure of shock waves. The grid resolution in the vicinity of the triple shock wave intersection is demonstrated in Fig. 5 (more than 10 cells inside the shock wave front for NS computations and more than 30 cells for DSMC computations). Here and below, $X$ and $Y$ are non-dimensionalized by the wedge length $w$. A sequence of grids was utilized to perform a grid

![Fig. 10](image-url)

**Fig. 10.** Numerical data vs shock polars in ($\theta, p$)-plane for different wedge angles. Numerical solution of Navier–Stokes equations at $M=1.7$, $\gamma=5/3$, $Re=2122.9$, and $Kn=0.001$. Points A, B, C, D, and E correspond to the points in the plane ($X, Y$) in Fig. 9. Case 1, wedge angle $\theta_w=9.63^\circ$ (a); Case 2, wedge angle $\theta_w=12^\circ$ (b); Case 3, wedge angle $\theta_w=13^\circ$ (c); Case 4, von Neumann paradox conditions, wedge angle $\theta_w=13.5^\circ$ (d); Case 4, enlarged view of the non-Rankine–Hugoniot region at two different Reynolds numbers (e).
resolution study. It was found that the results were grid independent if the value of the Reynolds number based on the grid cell size was less than 3. The issues of numerical accuracy of simulation results are discussed in Appendix II.

To eliminate possible DSMC and Navier–Stokes cross-validation issues, a monatomic gas, argon, with $\gamma=5/3$ is considered (DSMC modeling of diatomic gas flow always accounts for relaxation phenomena, which can significantly affect the internal structure of shock waves and the zone of their interaction), except for the comparison with the model developed in [26] (Sternberg’s case) where $\gamma=1.4$. The power-law dependence of the dynamic viscosity coefficient $\mu$ on temperature $T$ with the exponent $\omega=0.81$ was used in the Navier–Stokes computations. The variable hard sphere model 10 (VHS) was used in the DSMC computations with the VHS parameter chosen to provide the same dependence of viscosity on temperature. The relation between Reynolds and Knudsen numbers can be obtained using the formula for the mean free path $l$ in an equilibrium gas given in [29]. For the VHS model of intermolecular collisions, $l$ can be written as

$$l = \frac{2(5-2\omega)(7-2\omega)}{15\pi^{1/2}} \frac{\mu}{(2RT)^{1/2} \rho},$$

where $R$ is the gas constant, $T$ is the temperature, and $\rho$ is the density. Then, the relation between $Kn$ and $Re$ is

$$Kn = \frac{2(5-2\omega)(7-2\omega)}{15\pi^{1/2}} \left(\frac{\gamma}{2}\right)^{1/2} \frac{M}{Re}.$$

The Reynolds and Knudsen numbers were varied in the course of our computational study.

### 4. Weak irregular reflection for different wedge angles

The results of the Navier–Stokes computations for Case 1 are illustrated by Mach number flowfield in Fig. 6. If not indicated specifically, hereinafter we consider the computations for $Re=2122.9$ ($Kn=0.001$). The resultant overall shock wave configuration resembles a typical MR pattern. A closed subsonic region is formed behind the Mach stem and the reflected shock. The size of this region is governed by the geometrical parameter $g/w$, which is $g/w=1.5$ in this case. Note, in the case considered, the reflected shock wave is curved due to the influence of expansion waves that cause a pressure drop in the subsonic flow behind the reflected shock. The flow portions passing through the Mach stem and the reflected shock are separated by the mixing layer. The triple point appears as a smooth continuous transition zone from the Mach stem to the reflected shock. A detailed structure of this smooth three shock transition zone is shown as zoomed NS flowfields of the density and Mach number in Fig. 7. Note, in the rest of the paper, the gas dynamic parameters such as density, temperature, pressure are normalized by their free-stream values. The black lines in the figures show the inviscid three-shock solution: the angles of the shock waves and the slip surface to the incoming flow are uniquely determined from the three-shock theory; the triple point is located at the centre of the smooth three shock transition zone. It is clearly seen here that
isolines representing the Mach stem and reflected shock do not align with the slopes prescribed by the inviscid three-shock solution even near the triple point. Note, that in our previous studies of Mach reflection of strong shock waves (see e.g. [28]) the computations always reproduced the shock angles according to the three-shock solution with good accuracy. An interesting feature of the density flowfield is the local maximum behind the smooth three shock transition zone (see Fig. 7a, closed isoline 1.98). Note also a saddle point in the density distribution behind the Mach stem. Fig. 8 shows the overall density fields for all examined cases of weak reflection of shock waves. The flow pattern in all other cases coincides qualitatively with the usual Mach reflection. For all cases where the three-shock solution exists ($\theta_w=9.63^\circ, 12^\circ$, and $13^\circ$), the angle of inclination of the reflected wave diverges significantly from that predicted by the three-shock theory.

Fig. 9 shows the Mach number fields. The black solid curve shows the sonic line (with a subsonic flow inside and a supersonic flow outside). In all cases, the sonic line geometry is qualitatively similar far from the triple-shock-wave intersection region, regardless whether the three-shock solution exists or not. For Case 3, the inviscid three-shock theory predicts a supersonic flow immediately behind the reflected wave, but the flow behind the smooth three shock transition zone is subsonic due to viscous interaction in our low-Reynolds number simulations. Note that the flowfields for the higher Reynolds number Re=4246 ($Kn=0.0005$) are almost identical, except for a smaller thickness of shock waves.

The results of our viscous computations are compared with inviscid shock polar solutions in the plane ($\theta, p$) in Fig. 10 (the method of transposition of numerical results to the plane ($\theta, p$) is described in Appendix I). We select several characteristic points along the Mach stem and the reflected shock wave. The points lie near the rear surface of the shock wave beyond the steep gradients associated with the viscous internal structure of the shock waves. Points A, B, C, D, and E in the plane ($x, y$) (in Fig. 9) correspond to the points in the plane ($\theta, p$) in Fig. 10. Point A is located on the line of symmetry, where the Mach stem is a normal shock wave. Upward along the Mach stem (from point A to point B), the pressure decreases, and the angle of flow deflection increases, which is caused by the curved form of the Mach stem. At point B, the numerical data start to deviate from the theoretical polar. Point C corresponds to the maximum angle of flow deflection. Further motion along the reflected wave leads to a decrease in pressure and the flow deflection angle, with a considerable part of the reflected wave lying outside the reflected wave polar (CD): at a fixed angle of flow deflection, the pressure exceeds the theoretical value. At point D, the numerical data arrive on the reflected wave polar and coincide with the polar up to point E. Starting from point E, the reflected wave starts to interact with the expansion fan emanating from the trailing edge of the wedge.

Thus, Points A, B, C, D, and E divide the vicinity of the triple point into the following zones:

1. **AB**—Mach stem, this is the «conventional» shock wave, which obeys the Rankine–Hugoniot conditions;
2. **BCD**—zone of the transition from the Mach stem to the reflected wave, this is the zone where viscous effects cause the solution to diverge from the Rankine–Hugoniot conditions. In the planes ($\theta, p$), ($U, V$), ($\theta, r$), and ($\theta, T$), the numerical values from point B to point D do not lie on any of the polars (the planes ($U, V$), ($\theta, r$), and ($\theta, T$) are shown in the figures below);
3. **DE**—reflected wave, which obeys the Rankine–Hugoniot conditions, but the parameters behind this wave differ from those predicted by the three-shock theory;

![Fig. 13](image)

*Fig. 13. Temperature field in the vicinity of the triple point for Case 3. Numerical solution of Navier–Stokes equations, $M=1.7$, $Kn=0.001$, $\gamma=5/3$, and $Re=2122.9$.* Black solid curves show the shock wave locations predicted by the three-shock theory. The dashed lines are the sections where the distributions of the flow parameters were extracted.

![Fig. 14](image)

*Fig. 14. Sectional distributions of flow parameters for Case 3: pressure (a) and temperature (b). The dotted lines are the values corresponding to the inviscid three-shock solution.*
4. from point E and further downstream—zone of influence of the expansion fan.

The behavior of numerical data for all cases is qualitatively similar and does not depend on the type of intersection of the shock polars. The numerical data pass from the incident wave polar to the reflected wave polar along a curve that does not lie on the polars, i.e., through a zone where viscous effects cause the solution to diverge from the Rankine–Hugoniot conditions. Thus, the existence of such a finite transition zone (BCD) in the viscous case ensures a continuous transition of the gas-dynamic parameters from the incident wave polar to the reflected wave polar. Actually, the BCD zone is a “non-Rankine–Hugoniot zone” (see [26]). Note that the final result is independent of the choice of the criterion for constructing numerical data in the planes (U, V), (θ, p), (θ, ρ), and (θ, T) (see Appendix I). The comparison of our computations at two different Reynolds numbers presented for Case 4 (i.e., at von Neumann paradox conditions) demonstrate slight variations in pressure and flow deflection and do not demonstrate any particular trend with increasing the Reynolds number. The computations at higher Re would be very useful for understanding the asymptotic behavior of the Navier–Stokes solutions at Re → ∞.

The numerical results are further compared with the three-shock theory in more detail for cases 1, 3 where the inviscid three-shock solution exists. Fig. 11 shows the temperature field in the vicinity of the triple point for Case 1. As was already noted, the slope of the reflected wave differs from the slope predicted by the three-shock theory. The dashed lines indicate the sections in which the parameters plotted in Fig. 12 were taken. The pressure in the sections Y=0.22 and 0.24 does not reach the value predicted by the three-shock theory, and the pressure in the sections Y=0.18 and 0.2 exceeds the theoretical value by 2.5%. The temperature in the section Y=0.22 agrees well with the predicted theoretical value behind the reflected wave. Note that the temperature in the section Y=0.2 reaches the theoretical value behind the Mach stem, and the temperature in the section Y=0.18 exceeds this value.

Fig. 13 shows the temperature field and the sections of parameters in the vicinity of the triple point for Case 3. The parameters in these sections are plotted in Fig. 14. In the vicinity of the triple point, the pressure and temperature behind the reflected shock wave and the Mach stem is higher than the theoretical prediction (i.e., the Rankine–Hugoniot value). Note that a qualitatively similar result is also observed for Case 2. The difference confirms the conclusion about the existence of the non-Rankine–Hugoniot zone. Note that similar results are also obtained for the higher Reynolds number Re=4246 (Kn=5 × 10^-4).

Fig. 15a shows the density field in the vicinity of the triple point and the polars in the planes (U, V), (θ, ρ), and (θ, T) in comparison with numerical data for the case of the right subsonic intersection. The behavior of numerical data in the plane (θ, ρ) (Fig. 15c) is similar to the behavior of numerical data in the plane (θ, p), except for the segment BC, which contains a local density maximum. On the segment CD, there is a region where the density is higher than the density behind the normal shock (with respect to the flow behind the incident shock); this fact is clearly seen in
the plane \((\theta, \rho)\) (Fig. 15c). The local maximum of density in the vicinity of the triple point in the plane \((X, Y)\) is also clearly visible in Fig. 15a (closed isoline 1.93). It is seen in Figs. 15b, d that the behavior of numerical data in the planes \((\theta, T)\) and \((U, V)\) is similar to the behavior of numerical data in the plane \((\theta, \rho)\). The temperature reaches the theoretical values on the segments CD and DE, but the angle of flow deflection is smaller than the predicted value. In the hodograph plane, it is well seen that the \(y\)-component of velocity does not reach the theoretical predictions on the entire segment BD.

Fig. 16 shows the correspondence between the plane \((X, Y)\) and the planes \((U, V)\), \((\theta, \rho)\), and \((\theta, T)\) for the von Neumann paradox conditions (Case 4). As in the previous case, there is a local density maximum in the vicinity of the triple point (Fig. 16a), which is also visible in the plane \((\theta, \rho)\) (Fig. 16c). In the hodograph plane (Fig. 16b), the \(y\)-component of velocity does not reach the theoretical values, as in the previous case. Note that the behavior of numerical data (Figs. 16b–d) in the case of the von Neumann paradox is qualitatively similar to the behavior of numerical data for the cases where the three-shock solution exists.

---

Fig. 16. Numerical solution of Navier–Stokes equations. Case 4: \(M=1.7\), \(\theta_w=13.5\), \(\gamma=5/3\), Re=4246 \((Kn=5 \times 10^{-4})\). Density field in the vicinity of the triple point (a), hodograph plane (b), plane \((\theta, \rho)\) (c), plane \((\theta, T)\) (d). Points A, B, C, D, and E in the plane \((X, Y)\) correspond to the points in the planes \((U, V)\), \((\theta, \rho)\), and \((\theta, T)\).

Fig. 17. Flowfields in the smooth three shock transition zone: pressure (a) and Mach number (b) contours. DSMC computations, Case 1: \(M=1.7\), \(\gamma=5/3\), and \(\theta_w=9.63\). \(Kn=5 \times 10^{-4}\), Re=4246 (solid lines); \(Kn=0.001\), Re=2123 (dashed lines).
It is well known that the viscous spatial scales (including the shock wave thickness) are determined by the mean free path of molecules in the gas. If we assume that the transition zone structure is determined by viscous effects, we can expect that the transition zone size decreases with decreasing mean free path of molecules in the gas. Here, we can find an analogy with the viscous shock transition, where the shock wave thickness and its internal structure are independent of the Reynolds number in coordinates normalized to the free-stream molecular mean free path \( l \). To analyze the influence of viscosity on the flow structure in the smooth three shock transition zone, we compared the flowfields at different Reynolds numbers in normalized coordinates \( x/l, y/l \).

The results of numerical simulations for different Reynolds numbers are compared in Figs. 17–19. The flowfields for \( Re=2123 \) and 4246 are shifted in both coordinate directions to have the vicinities of shock wave intersection at one place. The isolines for different Reynolds numbers are very close to each other for the cases where the inviscid three-shock solution exist, as well as for the von Neumann paradox case. Thus, the flowfields in the vicinity of shock wave intersection for different Reynolds numbers are indeed geometrically similar in physical coordinates.

The case of the von Neumann paradox is considered in more detail in the next section. The results obtained with the use of the approximate viscous model of [26] of the flow in the smooth three shock transition zone are compared with numerical simulations.

5. Comparison of computational results with the Sternberg model

Let us briefly describe the approximate flow model in the vicinity of the triple point, which was proposed in [26]. The conservation equations for the buffer zone presented in [26] in the following form:

\[
\begin{align*}
(1) & \quad \text{Conservation of mass} \\
& \quad \rho_1 q_1 H_1 = \rho_2 q_2 (H_1 - \Delta T) \\
(2) & \quad \text{Conservation of x momentum component} \\
& \quad p_2 (H_1 - \Delta T) - p_1 H_1 + (F_x)_x + (F_p)_x = \rho_1 q_1 H_1 - \rho_2 q_2 (H_1 - \Delta T) \\
& \quad \text{y-component} \\
& \quad (F_{xy}) + (F_p)_y - p_2 (H_1 - \Delta T) \eta = \rho_2 q_2^2 (H_1 - \Delta T) \eta \\
(3) & \quad \text{Conservation of energy} \\
& \quad \left( \frac{q_2^2}{2} + C_v T_1 \right) \rho_1 q_1 H_1 = \left( \frac{q_1^2}{2} + C_v T_2 \right) \rho_2 q_2 (H_1 - \Delta T) \\
& \quad + p_2 (H_1 - \Delta T) q_2 - p_1 q_1 H_1 - Q - W .
\end{align*}
\]
In these equations, $H_1$ is the height of the domain where the Rankine–Hugoniot relations do not apply, $\rho_1$, $p_1$, $q_1$, and $T_1$ are the free-stream density, pressure, velocity, and temperature, $\rho_2$, $p_2$, $q_2$, and $T_2$ are the density, pressure, velocity, and temperature on the boundary $CD$ (see Fig. 20). Note that $H_1$ is understood as the product of the height and the unit length in the $z$ direction for the reasons of a correct dimension. The flow that passed through the buffer zone is assumed to deflect at a small angle, i.e., $\sin \theta \approx \theta$, $\cos \theta \approx 1$. $F_q$ is the force due to normal viscous stress on $GE$ and $BC$, $F_p$ is the force due to mean normal pressure on $GD$ and $AC$, $W_2$ is the work done on fluid in non-R–H region by shear stress, and $Q$ is the net heat transferred into non-R–H region through $GE$ and $BC$.

It is necessary to note that this system of conservation equations is not closed. To solve this system of equations, Sternberg used available experimental data to choose appropriate values of pressure and flow deflection angle at points C and E (for more details see [26]). This model allows one to obtain the estimates of gas-dynamic parameters ($\rho_2$, $p_2$, $q_2$, $M_2$) on the boundary $CD$. The height of this buffer zone was estimated as a few shock wave thicknesses.

It is impossible to confirm or reject the existence of such a viscous zone via inspection of the experimental data because of the insufficient resolution of the flowfield in the experiment. For comparisons of results, Sternberg [26] believed it would be desirable to perform experiments in a low-density wind tunnel. Unfortunately, such experiments have not been performed until now. At the moment, however, it is possible to simulate numerically the interaction of weak shock waves under the von Neumann paradox conditions on the basis of the Navier–Stokes or Boltzmann equations.

Further, we consider the details of the solution of Navier–Stokes equations for parameters estimated by Sternberg by the model for the triple shock intersection of [26] ($M=1.265$, $\theta_w=4.055^\circ$, $\gamma=1.4$). Under these conditions, the polars of the incident and reflected waves do not intersect, i.e., this case corresponds to the von Neumann paradox. Fig. 21 shows the results of the numerical solution of Navier–Stokes equations for the Reynolds number $\text{Re}=2000$. The flowfield qualitatively agrees with the results for $M=1.7$. The behavior of numerical results in the plane $(\theta, p)$ is also similar to the previously considered cases. The dotted lines are the results predicted by Sternberg’s approximate theory. It is clearly seen that the results predicted by the model for the triple shock intersection agree well with numerical data in the plane $(\theta, p)$. At point C, the pressure coincides with the value predicted by the model for the triple shock intersection, and the angle of flow deflection differs by 0.25°. The NS results for different Reynolds numbers are compared in Fig. 22 in the planes $(U, V)$, $(\theta, p)$, $(\theta, r)$, and $(\theta, T)$. It is seen that an increase in the Reynolds number leads to an increase in the deflection angle inside the smooth three shock transition zone. The results predicted by the Sternberg model are in good agreement with NS results for all parameters presented here.

6. Conclusion

A steady flow around symmetric wedges with a Mach number $M \sim 1.2–1.7$ and Reynolds numbers $\text{Re} \sim 1000–5000$ was numerically studied for conditions where the inviscid theory admits the existence of the three-shock solution as well as for conditions
where the three-shock solution does not exist (von Neumann paradox conditions). Our simulations performed with substantially different approaches suggest that the flow viscosity induces the formation of a smooth three shock transition zone, where one-dimensional Rankine–Hugoniot shock jump relations cannot be applied. The flow parameters in this zone differ from the theoretical values predicted by the inviscid three-shock solution. In particular, the maximum angle of flow deflection in the smooth three shock transition zone is smaller than its theoretical value.

Fig. 22. Comparison of calculations at different Reynolds numbers. The dotted lines are the result of Sternberg theory. Plane $(\theta, p)$ (a), hodograph plane (b), plane $(\theta, \rho)$ (c), plane $(\theta, T)$ (d).

Fig. 23. Numerical solution of Navier–Stokes equations, $M=1.7$, $\theta_0=9.63^\circ$, $\gamma=5/3$, $Re=2122.9$ (Kn=0.001). Black solid curve is the pressure isoline $dp/dx=4$.

Fig. 24. Derivative of the pressure with respect to the $x$ coordinate for $Y=0.1$. Numerical solution of Navier–Stokes equations, $M=1.7$, $\theta_0=9.63^\circ$, $\gamma=5/3$, and $Re=2122.9$ (Kn=0.001). The red solid curve shows the value corresponding to $dp/dx=4$. 
For the von Neumann paradox conditions, the computations predict an overall shock interaction configuration similar to Mach reflection. The existence of a viscous shock transition zone in the region of shock wave intersection allows a continuous transition from the parameters behind the Mach stem to the parameters behind the reflected shock, which is impossible in the inviscid three-shock theory.

The results of numerical simulations for von Neumann paradox conditions [26] confirmed the validity of the Sternberg approximate model and confirm the hypothesis about the existence of a transition zone where the Rankine–Hugoniot conditions do not apply.

The open question remains whether the viscous flow structure observed in this study can continuously transform into flow patterns predicted by the Guderley model of the vNR and recent inviscid numerical simulations [10,20,22] as the impact of dissipation decreases. The computations at much higher Reynolds numbers are required to clarify completely the behavior of the viscous solution.

**Acknowledgments**

This study was supported by the Russian Foundation for Basic Research (Grant no. 08-01-91307), Board of the Russian Academy of Sciences (Fundamental Research Program 11), Siberian Branch of the Russian Academy of Sciences (Integration of Research Project 26). The computations were performed at the Siberian Supercomputer Centre, Novosibirsk, and at the Joint Supercomputer Centre, Moscow.
Appendix I. Transfer of numerical data in the planes \((U, V), (\theta, p), (\theta, \rho),\) and \((\theta, T)\)

Numerical data in the planes \((U, V), (\theta, p), (\theta, \rho),\) and \((\theta, T)\) were taken along the nominal rear boundary of viscous shock waves (reflected shock and Mach stem). In the present study, the nominal boundary of shock waves was chosen to be an isoline of the pressure derivative with respect to the \(x\) coordinate (Fig. 23). The boundary was defined on the basis of a specified small value of this derivative as compared with its maximum value (Fig. 24). In the present study, we used the value \(dp/dx = 4\). Using the flow parameters at the points on this isoline \((x, y)\), we constructed the points in the planes \((U, V), (\theta, p), (\theta, \rho),\) and \((\theta, T)\) with the corresponding coordinates (e.g., see Fig. 25).

Note that the values of derivatives of other flow parameters can be used as a criterion for determining the isoline corresponding to the rear front of the shock wave; otherwise, this rear boundary can be constructed by the maximum pressure for a fixed \(y\) coordinate. The choice of the nominal rear boundary of shock waves, however, does not affect the final result in the plane \((\theta, p)\). Let us consider the field of the flow deflection angles (Fig. 26). The three-shock theory predicts that the angle of flow deflection behind the point should be 9.63°, but it is clearly seen in the figure that such values of the flow deflection angle are not observed behind the reflected shock and Mach stem. The shock polars in the plane \((\theta, p)\) are compared in Fig. 27 with numerical data constructed with different criteria used to choose the rear boundary of shock waves: based on small values of the isolines of the pressure and density derivatives, and also based on the maximum pressure with a fixed \(y\) coordinate. Good qualitative and quantitative agreement is observed for all criteria considered. Note that a similar behavior of numerical data is observed in all the considered cases of reflection of weak shock waves.

Appendix II. Numerical accuracy

The analysis of accuracy of DSMC computations is complicated by the presence of various errors, which may be classified as follows:

1. errors associated with discretization in time and space;
2. errors associated with a finite number of model particles in the system;
3. statistical errors of macroparameter averaging.

The statistical error of averaging is inversely proportional to the square root from the sampling magnitude. To increase the sampling, the results in the present study were averaged over time, beginning from the moment of flow stabilization. The statistical error in the results presented here is smaller than 1%.

The first two types of errors are usually analyzed by performing a series of computation with a decrease in the time step, grid refinement, and an increase in the number of model particles. Convergence of results of numerical simulations in terms of all these parameters shows that the errors are small and do not affect the final result. Such series of computations were performed in the present study. The convergence in terms of the number of model particles is demonstrated by pressure profiles along the line of symmetry \((y = 0)\) for different numbers of particles (Fig. 28).

It is seen in this figure that the results obtained with the use of 8.6 million particles differ from the results obtained with 34 million particles; the difference in the front position reaches almost 0.005 wedge lengths. With a further increase in the number of particles up to 70 million, the pressure remain unchanged, which allows us to conclude that the convergence in terms of the number of particles is reached. Note that obtaining this result took more than 24 h of computer time with 200 processors.

As was noted previously, the NS computations in the present study were performed with complete resolution of the internal structure of shock waves. In the results presented here, the resolution of the spatial grid used in solving Navier–Stokes equations was smaller than the local mean free path. In most cases, this fact ensures that numerical viscosity is small, as compared with physical viscosity, in solving Navier–Stokes equations.

To verify the numerical accuracy of results, we studied the convergence on a sequence of consecutively refined grids. Results in the plane \((\theta, p)\) and in the hodograph plane were found to be most sensitive to the resolution. Fig. 29 shows the results for the case of the right supersonic intersection (Case 3). It is seen that the computed results converge with increasing grid resolution. In particular, the results for the 1092 \(\times\) 984 and 2184 \(\times\) 1956 grid agree well with each other.

References


